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Mixed Type Partial Differential Equations With Initial and Boundary Values in Fluid Mechanics

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Abstract

This paper includes various parts of the theory of mixed type partial differential equations with initial and boundary conditions in fluid mechanics ,such as: The classical dynamical equation of mixed type due to Chaplygin (1904), regularity of solutions in the sense of Tricomi (1923) and in brief his fundamental idea leading to singular integral equations, and the new mixed type boundary value problems due to Gellerstedt (1935), Frankl (1945), Bitsadze and Lavrent'ev (1950), and Protter (1950-2007). Besides this work contains the classical energy integral method and quasi-regular solutions and weak solutions , as well as the well-posedness of the Tricomi , Frankl, and Bitsadze - Lavrent'ev problems in the sense that: "There is at most one quasi-regular solution and a weak solution exists". Furthermore Rassias (Ph.D. dissertation, U. C. , Berkeley, (1977)) generalized the Tricomi and Frankl problems in n dimensions based on Protter's proposal. This author generalizes even further the results obtained through the said thesis. Also this paper provides a maximum principle for the Cauchy problem of hyperbolic equations in multi-dimensional space-time regions, the formulation and solution of the Tricomi-Protter problem, a selection of several uniqueness and existence theorems and recent open problems suggested by Rassias in the theory of mixed type partial differential equations and systems with applications in fluid mechanics.

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Key Words & Phrases. Chaplygin equation, Tricomi problem, Gellerstedt problem, Frankl problem, Frankl condition, Bitsadze-Lavrent'ev problem, Rassias bi-hyperbolic equation, exterior Rassias-Tricomi problem, general Tricomi-Rassias (or GTR) problem, Tricomi-Protter problem.

1. INTRODUCTION

The theory of partial differential equations of mixed type with boundary conditions originated in the fundamental research of Tricomi [63]. The Mixed type partial differential equations are encountered in the theory of transonic flow and they give rise to special boundary value problems, called the Tricomi and Frankl problems. The Transonic flows involve a transition from the subsonic to the supersonic region through the sonic curve. Therefore, the transonic flows are very interesting phenomena appearing in aerodynamics and hydrodynamics. The well-known mixed type partial differential equation was called Tricomi equation : $yu_{xx} + u_{yy} = 0$ named after Tricomi ,who introduced this equation, for functions u=u(x, y) in a real (x, y)-region. It plays a central role in the mathematical analysis of the transonic flows, as it is of elliptic and hyperbolic type where the coefficient y of the second partial derivative of the involved function u=u(x, y) with respect to x , changes sign. Besides this equation is of parabolic type where y vanishes.

Definition 1. 1. The Tricomi problem or Problem T consists in finding a function u which satisfies the afore-mentioned Tricomi equation in a mixed domain D : a simply connected and bounded (x, y)-region by a rectifiable Jordan (non-self-intersecting) elliptic arc σ (for y positive) with endpoints O=(0,0) and A=(1,0) and by two real hyperbolic characteristics Γ, γ of the Tricomi equation satisfying the pertinent characteristic equation such that these characteristics Γ, γ meet at a point P (for y negative) with Γ emanating from A and γ from O,

$$\Gamma : x + \frac{2}{3}(-y)^{3/2} = 1 \quad \text{and} \quad \gamma : x - \frac{2}{3}(-y)^{3/2} = 0$$

and u assumes prescribed continuous boundary values on both arcs σ and γ. The portion of D lying in the upper half-plane, above the x-axis, is the elliptic region; portion of D lying in the lower half-plane, below the x-axis, is the hyperbolic region; and the segment OA is parabolic.

Definition 1. 2. A function u=u(x, y) is a regular solution of Problem T if:

1) u is continuous in the closure of D which is the union of D with its boundary consisting of the three curves σ, Γ, γ;

2) The first order partial derivatives of u are continuous in the closure of D except possibly at points O, A, where they may have poles of order less than 2/3;

3) The second order partial derivatives of u are continuous in D except possibly on OA where they may not exist;

4) u satisfies Tricomi equation at all points of D except OA;

5) u assumes prescribed continuous boundary values on arcs σ and γ.

Fundamental idea of Tricomi in finding Regular Solutions for Problem T:

1) To solve Neumann problem or Problem N (in elliptic sub-region of D): To find a regular solution of the Tricomi equation satisfying the boundary conditions:

1a) u assumes prescribed continuous boundary values on σ, such that, u equals to a continuous function ϕ on σ;

1b) The first order partial derivative of u with respect to y equals to a continuous function $\nu = \nu(x)$ on OA except possibly at points O, A, where it may go to infinity of order less than 2/3.

2) To solve Cauchy problem (in the hyperbolic sub-region of D):

To find a regular solution of the Tricomi equation satisfying the initial conditions:

2a) u assumes continuous values on OA such that u equals to a continuous function $\tau = \tau(x)$ on OA except possibly at points O, A, where it may go to infinity of order less than 2/3;

2b) The first order partial derivative of u with respect to y equals to a continuous function $\nu = \nu(x)$ on OA except possibly at points O, A, where it may go to infinity of order less than 2/3.

3) To solve Goursat problem (in the hyperbolic sub-region of D):

To find a regular solution of the Tricomi equation satisfying the boundary conditions:

3a) u assumes continuous values on OA such that u equals to a continuous function $\tau = \tau(x)$ on OA except possibly at points O, A, where it may go to infinity of order less than 2/3;

3b) u assumes prescribed continuous boundary values on γ such that u equals to a continuous function ψ on γ.

Most fundamental results on Mixed Type Equations with Applications:

In 1904, Chaplygin [2] has pointed out that the theory of gas flow is closely connected with the study of a mixed type equation named Chaplygin equation

$Lu := K(y)u_{xx} + u_{yy} = 0$

by replacing coefficient y of the second order partial derivative of u with respect to x in Tricomi equation by a function $K = K(y)$ of y . Consider a two-dimensional adiabatic potential flow of a perfect gas. The stream function $\psi = \psi(x, y)$ satisfies

$$(\rho^2\alpha^2 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (\rho^2\alpha^2 - \psi_x^2)\psi_{yy} = 0, \tag{1.1}$$

where: $\alpha :=$ the local velocity of sound and $\rho :=$ the density of the gas.

This equation is transformed to a linear equation of mixed type by applying the following hodograph transformation $u = \rho^{-1}\psi_y;\quad v = -\rho^{-1}\psi_x$,where u, v :=the rectangular velocity components as new independent variables.

The corresponding components in polar coordinates are: $r = \sqrt{u^2 + v^2};\quad \theta = \tan^{-1}(\frac{v}{u})$.

To normalize r , introduce : $t = (r / r_0)^2$, which is dimensionless quantity, as independent variable, where $r_0 :=$ the speed corresponding to zero density. Therefore above-mentioned equation (1.1) becomes

$$\frac{\partial}{\partial t}\left\{\frac{2t}{(1-t)^\beta}\psi_t\right\} + \frac{1-(1+2\beta)t}{2(1-t)^{\beta+1}}\psi_{\theta\theta} = 0, \tag{1.2}$$

where: $\beta = c_v / (c_p - c_v)$,such that: c_p :=the specific heat at a constant pressure,

$c_v :=$ the specific heat at a constant volume. Note that:

$r_0 = k\gamma\rho_0^{-1} / (\gamma - 1) : \gamma = c_p / c_v$, $\rho_0 :=$ the density of gas at zero speed,

k (:=constant) satisfies the relation: $p = k\rho^\gamma$.

Introduce new independent variables $\xi = \theta;\quad \eta = -\int_{\frac{1}{2\beta+1}}^{t} \frac{(1-u)^\beta}{2u} du$.

Then, the afore-mentioned equation (1.2) becomes

$$K(\eta)\psi_{\xi\xi} + \psi_{\eta\eta} = 0, \tag{1.3}$$

where: $K = K(\eta) = (1 - (1 + 2\beta)t).(1 - t)^{-(1+2\beta)}$, and

$\beta :=$ a positive constant ($\cong$ 2.5 for air) ; $\gamma :=$ a positive constant ($\cong$ 1.4 for air).

Besides: $K(0) = 0,\quad because\quad \eta = 0\ \ for\ \ t = 1/(2\beta + 1)$. This case corresponds to points where the velocity is equal to the local velocity of sound, and therefore above equation (1.3) is parabolic.

Moreover: $K(\eta) > 0$, because $\eta > 0\ \ for\ \ t < 1/(2\beta + 1)$;corresponding to subsonic velocities, and equation (1.3) is elliptic.

Similarly: $K(\eta) < 0$, because $\eta < 0\ \ for\ \ t > 1/(2\beta + 1)$;corresponding to supersonic velocities, and equation (1.3) is hyperbolic.

Therefore ,the partial differential equation (1.3) is of mixed type.

Remarks 1. 3.

i. The velocity potential $\phi = \phi(x, y)$ and the stream function $\psi = \psi(x, y)$ satisfy Cauchy-Riemann equations:

$\phi_x = \rho^{-1}\psi_y; \quad \phi_y = -\rho^{-1}\psi_x$.

ii. The discriminant of equation: $L\psi = 0$ is given by the formula

$D = (\rho^2\alpha^2 - \psi_y^2)(\rho^2\alpha^2 - \psi_x^2) - (\psi_x\psi_y)^2 = (\rho\alpha)^4(1 - M^2)$,

where: $M : Mach\ \ number := r / \alpha; \quad r = \sqrt{\psi_x^2 + \psi_y^2} / \rho$ .A flow is called subsonic, sonic or supersonic at a point as the flow speed $r\ < \alpha, = \alpha, or\ > \alpha.$

iii. Transonic flows involve a transition from the subsonic to the supersonic region through the sonic.

iv. Equation (1.1) is quasi-linear and is converted to the linear equation of mixed type (1.3). The corresponding equation is the following

quasi-linear equation: $(\alpha^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (\alpha^2 - \phi_y^2)\phi_{yy} = 0$,

where $\phi = \phi(x, y)$ is the velocity potential. This equation comes from

the Euler continuity equation: $(\rho\phi_x)_x + (\rho\phi_y)_y = 0$.

In 1935, Gellerstedt [8] generalized Problem T by replacing coefficient y of the second order partial derivative of u with respect to x in the Tricomi equation by a power of y. One may consider the well-known mixed type partial differential equation called Gellerstedt equation: $\mathrm{sg}\,n(y)\left|y\right|^m u_{xx} + u_{yy} = 0, m > 0$

named after S. Gellerstedt ,who introduced this equation.

Definition 1. 4. The Gellerstedt problem or Problem G consists in finding a function u which satisfies the afore-mentioned Gellerstedt equation in a mixed domain D : a simply connected and bounded (x, y)-region by a rectifiable Jordan (non-self-intersecting) elliptic arc σ (for y positive) with endpoints O=(0,0) and A=(1,0) and by two real hyperbolic characteristics Γ, γ of the Gellerstedt equation satisfying the pertinent characteristic equation such that these characteristics Γ, γ meet at a point P (for y negative) with Γ emanating from A and γ from O,

$$\Gamma : x + \frac{2}{m+2}(-y)^{(m+2)/2} = 1 \quad \text{and} \quad \gamma : x - \frac{2}{m+2}(-y)^{(m+2)/2} = 0$$

and u assumes prescribed continuous boundary values on both arcs σ and γ .

In 1945, Frankl [5] drew attention to the fact that the Tricomi problem is closely related to study of gas flow with nearly sonic speeds.

Definition 1. 5. The Frankl problem or Problem F consists in finding a function u which satisfies the afore-mentioned Chaplygin equation in a mixed domain D : a simply connected and bounded (x, y)-region by a rectifiable Jordan (non-self-intersecting) elliptic arc σ (for y positive) with endpoints O=(0,0) and A=(1,0) and by one real hyperbolic characteristic Γ (from A), $\Gamma : x = \int_0^y \sqrt{-K(t)}dt + 1,\ for\ y < 0$

of the Chaplygin equation satisfying the pertinent characteristic equation and by the non-characteristic curve γ' (emanating from O) lying inside the characteristic triangle OAP and intersecting the characteristic Γ at most once, where γ' may coincide with the hyperbolic characteristic curve $\gamma : x = -\int_0^y \sqrt{-K(t)}dt,\ for\ y < 0$, and Γ, γ intersect at point P as well as u assumes prescribed continuous boundary values on both arcs σ and γ'.

In 1950, Lavrent'ev and Bitsadze [14] initiated the work on boundary value problems for mixed type equations with discontinuous coefficients. One can consider the well-known mixed type Bitsadze - Lavrent'ev equation: $\operatorname{sgn}(y)u_{xx} + u_{yy} = 0,$ named after Lavrent'ev and Bitsadze, who introduced this equation with the discontinuous coefficient $\operatorname{sgn}(y) := 1, y > 0; = -1, y < 0; = 0, y = 0$.

Definition 1. 6. The Bitsadze - Lavrent'ev problem or Problem BL consists in finding a function u which satisfies the afore-mentioned Bitsadze - Lavrent'ev equation in a mixed domain D : a simply connected and bounded (x, y)-region by a rectifiable Jordan (non-self-intersecting) elliptic arc σ (for y positive) with endpoints O=(0,0) and A=(1,0) and by two real hyperbolic characteristics Γ, γ of :The Bitsadze - Lavrent'ev equation satisfying the pertinent characteristic equation such that these characteristics Γ, γ meet at a point P (for y negative) with Γ emanating from A and γ from O, $\Gamma : x - y = 1$ and $\gamma : x + y = 0$ and u assumes prescribed continuous boundary values on both arcs σ and γ.

The works performed in 1953, by Guderley [9] , Busemann [1], and Protter [16] are of great importance in the field of mixed type equations with applications. In particular, Protter established certain uniqueness theorems for boundary value problems involving equations of mixed elliptic-hyperbolic type by improving the famous Frankl condition $F(y) = 1 + 2(\frac{K(y)}{K'(y)})' > 0,\ for\ y < 0$.

In 1954, Protter [17]showed how problems of Goursat type other than those considered by Soboleff [62] and Garding [7] may be stated and solved for the multi-dimensional wave equation. These results lead in a natural way to certain generalizations of the Tricomi problem for equations of mixed elliptic-hyperbolic type. Protter introduced the following definition:

Definition 1. 7. A function u=u(x, y) is quasi-regular solution of Problem T if:

1) $u \in C^2(D) \cap C(\bar{D}), \bar{D} = D \cup \partial D, \partial D = \sigma \cup \Gamma \cup \gamma$;

2) The integrals $\int_0^1 u(x,0)u_y(x,0)dx \quad and \quad \iint_{D\cap\{y>o\}} (Ku_x^2 + u_y^2)dxdy$ exist;

3) Green's theorem is applicable to the integrals

$$\iint_D uLudxdy,\ \iint_D u_x Ludxdy,\ \iint_D u_y Ludxdy;$$

4) The boundary integrals which arise exist in the sense that: The limits taken over corresponding interior curves exist as these interior curves approach the boundary;

5) u satisfies Chaplygin equation in D;

6) u satisfies prescribed continuous boundary conditions:

$$u = \phi(s)\ on\ \sigma\ ;\ and\ u = \psi(x)\ on\ \Gamma.$$

2. UNIQUENESS AND EXISTENCE OF SOLUTIONS

In 1977, Rassias [20] generalized the Tricomi and Frankl problems in n dimensions, worked the hyperbolic degenerate boundary value problem in n dimensions, as well, and introduced a new boundary value problem in the theory of mixed type equations:

"The bi-hyperbolic degenerate problem" for the bi-hyperbolic degenerate equation:

$Lu = K(z)(u_{xx} - u_{yy}) + u_{zz} + \lambda(x,y,z)u = f(x,y,z)$ such that:

$K(z) > 0\ for\ z > 0; = 0\ for\ z = 0; < 0\ for\ z < 0$ and $\lambda \le 0$; $\lambda_x - \lambda_y \ge 0$,

in $\{(x,y,z)\}$ - bounded simply-connected region $D(\subset \mathbb{R}^3)$ surrounded:

for $z > 0$ by a characteristic surface S_3^y and a smooth surface Σ_4^y:

$$S_3^y : y + x_0 = \left[x^2 + \left(\int_0^z \sqrt{K(t)}dt \right)^2 \right]^{1/2} ;\ \Sigma_4^y : y - x_0 = -\left[x^2 + \left(\rho \int_0^z \sqrt{K(t)}dt \right)^2 \right]^{1/2}$$

and for $z < 0$ by a new characteristic surface S_4^x and a smooth surface Σ_3^x:

$$S_4^x : x - x_0 = -\left[y^2 + \left(\int_0^z \sqrt{-K(t)}dt \right)^2 \right]^{1/2} ;\ \Sigma_3^x : x + x_0 = \left[y^2 + \left(\rho \int_0^z \sqrt{-K(t)}dt \right)^2 \right]^{1/2}$$

with non-negative real constants $x_0 > 0; \rho \ge 1$.

All these surfaces intersect the $\{(x,y)\}$ -plane at the straight lines:

$l_1 : x + y + x_0 = 0$; $l_2 : x + y - x_0 = 0$; $l_3 : x - y + x_0 = 0$; $l_4 : x - y - x_0 = 0$.

Then Rassias introduced the following mixed type boundary value problem:

The bi-hyperbolic degenerate boundary value problem in $\mathbb{R}^3$:A solution $u = u(x,y,z)$ of the above bi-hyperbolic degenerate equation in D is found assuming prescribed boundary values on $\Sigma_3^x \cup \Sigma_4^y$, such that

$u = 0\ on\ \Sigma_3^x \cup \Sigma_4^y$ and $u \in C^2(\bar{D})$; $K \in C^2(\bar{D})$; $\lambda \in C^1(\bar{D})$; $f \in C^0(\bar{D})$,

where $\bar{D} = D \cup \partial D$,and $\partial D = S_4^x \cup S_3^y \cup \Sigma_3^x \cup \Sigma_4^y$ is the boundary of domain D.

Definition 2. 1. A function $u = u(x,y,z)$ is called quasi-regular solution of the above bi-hyperbolic degenerate problem if the following conditions hold:

1) $u \in C^2(D)$; 2) The integral $\int_{D \cap \{z=0\}} [.] dxdy$ exists;

3) If $D_i(\varepsilon)\ (i = 1,2)$ are regions with boundaries $\partial D_i(\varepsilon)$ lying entirely in D_i such that $D_1 = D \cap \{z > 0\}$; $D_2 = D \cap \{z < 0\}$, the boundary integrals along $\partial D_i(\varepsilon)$,which result from the Green's theorem to the integrals

$$\iiint_{D_i(\varepsilon)} u_x dV \ ;\ \iiint_{D_i(\varepsilon)} u_y dV \quad (dV = dxdydz)$$

have a limit when $\partial D_i(\varepsilon)$ approaches ∂D_i .Then he stated and proved:

Uniqueness Theorem 2. 2. If we assume that $\lambda \le 0$; $\lambda_x - \lambda_y \ge 0$ in mixed domain D, this bi-hyperbolic degenerate problem has at most one quasi-regular solution in D.

In 1979 and 1982, Rassias ([21]-[27]) published the afore-mentioned part of his Ph. D. Dissertation (U. C. Berkeley, 1977) on uniqueness results for the Frankl problem for his bi-hyperbolic degenerate equation: $Lu = K(z)(u_{xx} - u_{yy}) + u_{zz} + \lambda(x,y,z)u = f(x,y,z)$

and an application of positive symmetric systems [30].

In 1980, Rassias [22] introduced the bi-hyperbolic boundary value problem for his new bi-hyperbolic equation: $Lu = K(y)u_{xx} + \operatorname{sgn}(y)u_{yy} + r(x,y)u = f(x,y)$.

In 1982, Rassias [28] established maximum principles of the Cauchy problem for hyperbolic equations in $\mathbb{R}^3$ and $\mathbb{R}^{n+1}$ $(n \ge 2)$ .His results generalized the results of Weinberger [64] and Sather ([59]-[60]) for a class of hyperbolic equations of the form: $Lu = K(t)(u_{x_1x_1} + u_{x_2x_2}) + u_{tt} = f(x,t) \in C^0(\bar{D})$, where $x = (x_1, x_2)$; $D = \left\{ (x,t) : T_0 \le t \le T < 0,\ x_1^2 + x_2^2 < \left(\int_0^T \sqrt{-K(s)}ds \right)^2 \right\}$; $\bar{D} = D \cup \partial D$,

in $\{x_1, x_2, t\}$ -space and analogously generalized in $\{x_1, x_2, ..., x_n, t\}$ - space.

Throughout this investigation, the influence of the work of Douglis [4] and of the famous book of Protter and Weinberger [19] is apparent. In fact, let S_0 be the portion of the boundary ∂D of D, which lies in the plane $t = T_0$, and S_4 be the remainder of the boundary G of D, which is a characteristic conoid with respect to the above hyperbolic equation. Then Rassias introduced a solution $u = u(x,t)$ of this equation in D which satisfies the initial (or Cauchy) conditions $u(x,T_0) = \bar{r}(x) \in C^2(\bar{D}), \quad u_t(x,T_0) = h(x) \in C^0(\bar{D}).$ With the change of variable $v(x,t) = u(x,t) - \bar{r}(x)$ the afore-mentioned hyperbolic equation can be written as follows: $Lv = K(t)(v_{x_1x_1} + v_{x_2x_2}) + v_{tt} = F(x,t) \in C^0(\bar{D}),$ where $F = F(x,t) = f(x,t) - K(t)(\bar{r}_{x_1x_1} + \bar{r}_{x_2x_2})$. The initial data above change to

$$v(x, \mathrm{T}_0) = 0 \quad v_t(x,T_0) = h(x) \in C^0(\bar{D}).$$

Assume the transformation to spherical polar coordinates:

$$x_1 = r\cos p, \quad x_2 = r\sin p, \text{ where } p \in [0, 2\pi), \quad r = \int_t^T \sqrt{-K(s)}ds.$$

Therefore Rassias obtained $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}, \quad \Delta v = v_{x_1x_1} + v_{x_2x_2} = v_{rr} + \dfrac{1}{r}v_r + \dfrac{1}{r^2}v_{pp},$

$$Lv = K(t)(v_{rr} + \frac{1}{r^2}v_{pp}) + \frac{1}{r}K(t)v_r + v_{tt} = F_0 = F_0(r,p,t) \in C^0(\bar{D}).$$

Let S_4^e be the truncated part of S_4 , where $r_0 > e,$ such that

$$r_0 = \int_{T_0}^T \sqrt{-K(s)}ds, \quad e = \int_{T^0}^T \sqrt{-K(s)}ds \ : T_0 \le t \le T^0 \le T < 0.$$

The direct characteristic conoid S_4 as well as the truncated one S_4^e are generated by the bi-characteristics of space-time: $x_1 = r\cos p, \quad x_2 = r\sin p.$

The angle parameter p is constant along each fixed generator of S_4. On the other hand, the total derivative of S_4 is given by $\dfrac{d}{dr} = -\dfrac{1}{\sqrt{-K(t)}}\dfrac{\partial}{\partial r} - \sqrt{-K(t)}\dfrac{\partial}{\partial r}.$

Setting $d^+ = \dfrac{\partial}{\partial t} + \sqrt{-K(t)}\dfrac{\partial}{\partial r}, \ w = w(r,p,t) = r^{1/2}(-K(t))^{-1/4}; \quad V = \int_{p=0}^{2\pi} v(r,p,t)dp \ \ on \ \ S_4^e,$

Rassias stated and proved the following maximum principle:

Maximum Principle 2. 3. Let us suppose that the function $v(r,p,t)\in C^2(\bar{D})$ satisfies the differential inequality $Lv=F_0\le 0,$ and inequality : $v(r,p,T_0)=0,\ v_t(r,p,T_0)=h(r,p)\le 0,$

where D is defined above, and assume the following condition:

$\frac{d}{dr}\left(w\frac{K'(t)}{2\sqrt{-K(t)}}\right)-\sqrt{-K(t)}\frac{d}{dr}\left(d^+(w)\right)\ge 0,$ and : $V=0$. Then $v(r,p,t)\le 0\ \ in\ \ D.$

In 1983, the author [31] introduced the bi-elliptic, hyperbolic and bi-parabolic equation: $Lu=y(y-1)u_{xx}+u_{yy}=0$ with two parabolic lines: $y=0$; $y=1$.

Most of the recent workers in the field of mixed type boundary value problems have considered only one parabolic line of degeneracy. The problem with two parabolic lines of degeneracy becomes more complicated. Rassias introduced an equation:

bi-elliptic for $y>1, and\ y<0$; hyperbolic for $0<y<1$; and

bi-parabolic for $y=0\ and\ y=1$.

Besides Rassias assumed D as a simply connected and bounded domain by:An elliptic curve Γ_0 emanating from the points $A_1(0,1)$ and $B_1(1,1)$ and lying in the upper half-plane $y>1$; Another elliptic curve Γ'_0 emanating from the points $A_2(0,0)$ and $B_2(1,0)$ and lying in the lower half-plane $y<0$; and

The following four characteristic hyperbolic curves of our above mixed type equation lying in the region $G_2=\{(x,y):x>0,0<y<1\}\subset\mathbb{R}^2$, such that:

$$\Gamma_1(=A_1P_1):x=-\int_1^y\sqrt{y(1-y)}\,dy=\frac{1}{4}(1-2y)\sqrt{y-y^2}-\frac{1}{8}\sin^{-1}(2y-1)+\frac{\pi}{16}\ ;$$

$$\Gamma'_1(=A_2P_1):x=\int_0^y\sqrt{y(1-y)}\,dy=\frac{1}{4}(2y-1)\sqrt{y-y^2}+\frac{1}{8}\sin^{-1}(2y-1)-\frac{3\pi}{16}\ ;$$

$$\Gamma_2(=B_1P_2):x=1+\int_1^y\sqrt{y(1-y)}\,dy=\frac{1}{4}(2y-1)\sqrt{y-y^2}+\frac{1}{8}\sin^{-1}(2y-1)+1-\frac{\pi}{16}\ ;$$

$$\Gamma'_2(=B_2P_2):x=1-\int_0^y\sqrt{y(1-y)}\,dy=\frac{1}{4}(1-2y)\sqrt{y-y^2}-\frac{1}{8}\sin^{-1}(2y-1)+1+\frac{3\pi}{16}.$$

If one considers the afore-mentioned mixed type equation, then Rassias introduces the Tricomi problem which consists in finding a function $u=u(x,y)$ that satisfies this equation and the boundary condition $u|_{\Gamma_0\cup\Gamma'_0\cup\Gamma_2\cup\Gamma'_2}=0$ .Then he stated and proved:

Uniqueness Theorem 2. 4. If one assumed the star-likedness conditions:

$$\begin{cases} xdy-(y-1)dx\ge 0 & on\quad \Gamma_0 \\ xdy-ydx\ge 0 & on\quad \Gamma'_0 \end{cases},$$

then the Tricomi problem has at most one quasi-regular solution in D.

In 1988, Rassias [37] introduced the non-linear elliptic, hyperbolic and parabolic equation: $\tilde{L}w=K(y)w_{xx}+w_{yy}=f(x,y,w,w_x,w_y)$ and proved three uniqueness theorems for quasi-regular and regular solutions $w=w(x,y)$ of the Tricomi problem.

In fact, he assumed the following conditions:

(c_1): K(0)=0;K′(y)>0 for y<0 ; $\lim_{y\to 0^-}\frac{K(y)}{K'(y)}=0$,

$(c_2):(p=w_x;q=w_y)$:are continuous functions of $x,y,w,p,q:w,p,q\in\mathbb{R}^1$,

for any x, y in a simply-connected domain G bounded for $y>0$ by a non-self-intersecting Jordan smooth curve Γ_1 intersecting the $x-axis\ (:y=0)$ at the points $O_1(0,0)$; $O_2(1,0)$, and for $y<0$ by the characteristic curves Γ_2 ; Γ_3 of the above non-linear equation emanating from the points $O_2(0,0)$; $O_1(1,0)$, respectively, and intersecting at some point in the lower half-plane, such that

$$\Gamma_2: x=1+\int_0^y\sqrt{-K(t)}dt \quad ; \quad \Gamma_3: x=-\int_0^y\sqrt{-K(t)}dt .$$

Denote $G_0=G\cap\{y=0\}$; $G_1=G\cap\{y>0\}$; $G_2=G\cap\{y<0\}$, and boundary

$\partial G=\Gamma_1\cup\Gamma_2\cup\Gamma_3$. The above-mentioned non-linear equation is elliptic in G_1,

hyperbolic in G_2 and parabolic in G_0. Also he assumed prescribed continuous boundary values: $w=g\ \ on\ \ \Gamma_1\cup\Gamma_2$, where $g=g(s_0);\ s_0\in\Gamma_1\cup\Gamma_2$, and g is continuous on $\Gamma_1\cup\Gamma_2$. The author assumed two solutions $w_1\ \ and\ \ w_2$.

Then $u=w_1-w_2$ satisfies equation : $Lu=K(y)u_{xx}+u_{yy}-\Delta f=0$, or

$$Lu=K(y)u_{xx}+u_{yy}-r(x,y)u-s(x,y)u_x-t(x,y)u_y=0, \tag{2.1}$$

where $r=r(x,y)=\int_{h=0}^{h=1}f_w(x,y,w_2+hu,w_{2x}+hu_x,w_{2y}+hu_y)dh$,

$s=s(x,y)=\int_{h=0}^{h=1}f_p(x,y,w_2+hu,w_{2x}+hu_x,w_{2y}+hu_y)dh$,

$t=t(x,y)=\int_{h=0}^{h=1}f_q(x,y,w_2+hu,w_{2x}+hu_x,w_{2y}+hu_y)dh$,

and $\Delta f=f(x,y,w_1,w_{1x},w_{1y})-f(x,y,w_2,w_{2x},w_{2y})$

$=f(x,y,w_2+u,w_{2x}+u_x,w_{2y}+u_y)-f(x,y,w_2,w_{2x},w_{2y})$

(because: $u=w_1-w_2,\ or\ \ w_1=w_2+u),\ or$

$\Delta f=\int_{h=0}^{h=1}df(x,y,w_2+hu,w_{2x}+hu_x,w_{2y}+hu_y)=ru+su_x+tu_y$. Besides

$$u=0\ \ on\ \ \Gamma_1\cup\Gamma_2. \tag{2.2}$$

Mixed type boundary value problem or Problem (M):
consists in finding a function $u=u(x,y)$ which satisfies equation (2.1) in a mixed type domain G and assumes boundary conditions (2.2). Denote: $\beta=\frac{\pi}{2}\frac{1}{y_m\sqrt{K(y_m)}}$, $\gamma=\frac{\pi}{2}\frac{1}{y_m}$:

$where\ \ y_m$:= maximum of the ordinates of points on Γ_1;

$f_0=\frac{1}{4}\frac{f_p^2+Kf_q^2+2K(K\beta^2-\gamma^2)}{K}$, $\quad K\beta^2-\gamma^2=(\sqrt{\frac{K(y)}{K(y_m)}}-1)\gamma^2<0$, as K is:

$y < y_m$ *implies* $K(y) < K(y_m)$; $R = R(y) = 1 + 2\left(\frac{K}{K'}\right)' - 2\frac{K\sqrt{-K}}{K'}\beta$ *in* G_2.

Assume the following additional conditions:

$(c_3):\quad f_w \ge f_0 \quad for \quad (x, y, w, p, q) \in G_1 \times \mathbb{R}^3$,

$(c_4):\quad R \ge 0 \quad in \quad G_2$,

$(c_5):\quad f_p = \sqrt{-K}\, f_q \quad for \quad (x, y, w, p, q) \in G_2 \times \mathbb{R}^3$,

$(c_6):\quad 0 \le f_p \le \frac{1}{4}\frac{K'}{\sqrt{-K}}R \qquad for \quad (x, y, w, p, q) \in G_2 \times \mathbb{R}^3$, and

$(c_7):\quad 4(-K)(K')^2 f_w R \ge (4(-K)^{\frac{3}{2}} f_w + K' f_p)^2,\ f_w \ge 0$, or

$f_{01} \le f_w \le f_{02} \qquad for \quad (x, y, w, p, q) \in G_2 \times \mathbb{R}^3$,

where $f_{01} = \frac{1}{8}\frac{K'}{K^2}\left[\left(K' R - 2\sqrt{-K}\, f_p\right) - \sqrt{K'\left(K' R - 4\sqrt{-K}\, f_p\right)R}\,\right]$, and

$f_{02} = \frac{1}{8}\frac{K'}{K^2}\left[\left(K' R - 2\sqrt{-K}\, f_p\right) + \sqrt{K'\left(K' R - 4\sqrt{-K}\, f_p\right)R}\,\right]$.

Special Case: $f_p = 0 \quad (therefore: f_q = 0): (c_3)':\quad f_w \ge \frac{1}{2}(K\beta^2 - \gamma^2)$ in $G_1 \times \mathbb{R}^1$,

$(c_7)':\quad 0 \le f_w \le \frac{1}{4}\left(\frac{K'}{K}\right)^2 R$ in $G_2 \times \mathbb{R}^1$.

Then Rassias stated and proved the following uniqueness theorem:

Uniqueness Theorem 2. 5. If one assumed $(c_i):(i = 1,2,3,4,5,6,7)$, then the above Problem (M) has at most one quasi-regular solution in domain G.

To prove the uniqueness of quasi-regular solutions and thus above theorem one may consider the differential operator $Mu = a(x, y)u + b(x, y)u_y + c(x, y)u_y$:

$$a = \begin{cases} -\frac{1}{2}e^{\beta x}\cos(\gamma y), in\ G_1 \\ -\frac{1}{2}e^{\beta x}, in\ G_2 \end{cases};\quad b = \begin{cases} 0 & , in\ G_1 \\ c\sqrt{-K} & , in\ G_2 \end{cases};\quad c = \begin{cases} 0 & , in\ G_1 \\ \frac{4aK}{K'} & , in\ G_2 \end{cases},$$

and Green's theorem on the double integral in G: $\langle Mu, Lu \rangle_G = \iint_G MuLudxdy$.

Besides to investigate regular solutions we consider above conditions in a new simply connected domain G_e (part of domain G) bounded in the neighborhood of O_1 by a circular arc D_e (y>0) with center O_1 and radius e, and near O_2 by S_e ($y > 0$) : S_e is the line: $x = 1 - e$, if $x < 1$ *on* Γ_1 near O_2; S_e consists of the two the lines: $x = 1 - e$; $y = \sqrt{e}$, if $x \ge 1$ *on* Γ_1 near O_2, and by characteristics:

$dx = -\sqrt{-K}\, dy$.

Denote: $G_{1e} = G_e \cap \{y > 0\}$, $G_{2e} = G_e \cap \{y < 0\}$, $G_{0e} = G_e \cap \{y = 0\}$, and boundaries:

$\partial G_{1e} = \Gamma_1 \cup D_e \cup G_{0e} \cup S_e$, $\partial G_{2e} = \Gamma_2 \cup \Gamma''_{3e} \cup G_{0e} \cup \Gamma'_{3e}$;
$\partial G_e = \Gamma_1 \cup \Gamma_2 \cup D_e \cup \Gamma'_{3e} \cup \Gamma''_{3e} \cup S_e$.

Assume the following regularity condition: For some constant c_1,

$$(c_8): \quad K'(y)/\sqrt{-K(y)} \le c_1(-y)^{\alpha-1} \quad on \quad \Gamma_3 \quad near \quad O_1; \alpha > 0, c_1 > 0.$$

Definition 2. 6. A solution $u = u(x, y)$ of Problem (M) is called regular solution if it (1). satisfies equation (2.1) in G *except* G_0;

(2). is continuous in G;

(3). has continuous first derivatives in G and Γ_1, Γ_2 (*both open*) with the possible exception of the points O_1, O_2, in whose neighborhoods they may have poles of order less than 1: $u_x, u_y = o\left(|O_iP|^{-\omega_i}\right), 0 \le \omega_i < 1;\ |O_iP| \to 0\ (i = 1,2), P \in G,$ *and*

(4). has continuous second derivatives in G_i $(i = 1,2)$ with the possible exception of points on the parabolic curve, in whose neighborhoods they may not exist.

Uniqueness Theorem 2. 7. If one assumed conditions $(c_i):(i = 1,2,3,4,5,6,7,8)$, then the above Problem (M) has at most one regular solution in domain G.

In 1990, Rassias [39] introduced the bi-elliptic, bi-hyperbolic and bi-parabolic equation: $Lu = K(y)u_{xx} + u_{yy} + r(x, y)u = f(x, y)$ with two parabolic lines: $y = 0$, $y = 1$ .These boundary value problems with boundary conditions on the exterior boundary of a mixed type doubly-connected domain generalize his above uniqueness results [31]. He noted that this doubly-connected domain $D = G_1 \cup G'_1 \cup G_2 \cup G'_2$ consisted of the following four regions:

$G_1 (\subset R_1 = \{(x, y): |x| < 1, y > 1\})$:"upper elliptic region" ,

$G'_1 (\subset R'_1 = \{(x, y): |x| < 1, y < 0\})$:"lower elliptic region" ,

$G_2 (\subset R_2 = \{(x, y): 0 < x < 1, 0 < y < 1\})$:"right hyperbolic region" ,

$G'_2 (\subset R'_2 = \{(x, y): -1 < x < 0, 0 < y < 1\})$:"left hyperbolic region",

with "boundaries": $\partial G_1 = \Gamma_0 \cup A_1B_1$, $\partial G'_1 = \Gamma'_0 \cup A_2B_2$,

$\partial G_2 = \Gamma_1 \cup \Gamma'_1 \cup \Gamma_2 \cup \Gamma'_2 \cup O_1B_1 \cup O_2B_2$, $\partial G'_2 = \Delta_1 \cup \Delta'_1 \cup \Delta_2 \cup \Delta'_2 \cup O_1A_1 \cup O_2A_2$;

"edge" points: $O_1 = (0,1), O_2 = (0,0), A_1 = (-1,1), A_2 = (-1,0), B_1 = (1,1), B_2 = (1,0)$;

"parabolic" segments: $A_1B_1 = \{(x, y): |x| < 1, y = 1\}$:"upper parabolic segment",

$A_2B_2 = \{(x, y): |x| < 1, y = 0\}$:"lower parabolic segment",

$O_1B_1 = \{(x, y): 0 < x < 1, y = 1\}$:"right upper parabolic segment",

$O_2B_2 = \{(x, y): 0 < x < 1, y = 0\}$:"right lower parabolic segment",

$O_1A_1 = \{(x,y): -1 < x < 0, y = 1\}$:"left upper parabolic segment",

$O_2A_2 = \{(x,y): -1 < x < 0, y = 0\}$:"left lower parabolic segment" ;

"elliptic star-liked " arcs: Γ_0 :"connecting points A_1, B_1 in G_1", Γ'_0 :"connecting points A_2, B_2 in G'_1"; and eight "hyperbolic characteristic" lines of the above bi-hyperbolic equation:

$\Gamma_1(= O_1P_1): x = -\int_1^y \sqrt{-K(t)}dt$, $\Gamma'_1(= O_2P_1): x = \int_0^y \sqrt{-K(t)}dt$;

$\Gamma_2(= B_1P_2): x = 1 + \int_1^y \sqrt{-K(t)}dt$, $\Gamma'_2(= B_2P_2): x = -\int_1^y \sqrt{-K(t)}dt$,

where $P_1 = \Gamma_1 \cap \Gamma'_1$ and $P_2 = \Gamma_2 \cap \Gamma'_2$, as well as:

$\Delta_1(= A_1P'_1): x = -1 - \int_1^y \sqrt{-K(t)}dt$, $\Delta'_1(= A_2P'_1): x = -1 + \int_0^y \sqrt{-K(t)}dt$;

$\Delta_2(= O_1P'_2): x = \int_1^y \sqrt{-K(t)}dt$, $\Delta'_2(= O_2P'_2): x = -\int_0^y \sqrt{-K(t)}dt$,

where $P'_1 = \Delta_1 \cap \Delta'_1$ and $P'_2 = \Delta_2 \cap \Delta'_2$. Then Rassias stated and proved:

Uniqueness Theorem 2. 8. If one assumed the star-likedness conditions:

$$\begin{cases} xdy - (y-1)dx \geq 0 \quad on \quad \Gamma_0 \\ xdy - ydx \geq 0 \quad on \quad \Gamma'_0 \end{cases},$$

and the coefficient conditions:

$$\begin{cases} 2r + xr_x + (y-1)r_y \leq 0 \;\; in \;\; G_1 \\ r + xr_x \leq 0 \;\; in \;\; G_2 \cup G'_2 \\ 2r + xr_x + yr_y \leq 0 \;\; in \;\; G'_1 \end{cases} ; \; r \leq 0 \;\; on \;\; Int(D); K' > 0 \;\; in \;\; G_1 \; ; \; K' < 0 \;\; in \;\; G'_1 ,$$

where $Int(D) = \Gamma_1 \cup \Gamma'_1 \cup \Delta_2 \cup \Delta'_2$ (interior of D), as well as boundary conditions: $u = 0 \;\; on \;\; Ext(D)$,where $Ext(D) = \Gamma_0 \cup \Gamma'_0 \cup \Gamma_2 \cup \Gamma'_2 \cup \Delta_1 \cup \Delta'_1$ (exterior of D), such that: $\partial D = Ext(D) \cup Int(D)$ (boundary of D), then this Tricomi problem would have at most one quasi-regular solution in D.

Similarly Rassias investigated the pertinent Frankl problem. Besides he considered the following interesting special case:

$K = K(y) = \mathrm{sgn}(y(y-1))|y|^{\alpha}|y-1|^{\beta} k(y)$ ($\alpha > 0, \beta > 0 ; k = k(y) > 0$) in D.

Besides Rassias [33] (in 1985) investigated the pertinent Bitsadze-Lavrent'ev problem with two parabolic lines of degeneracy and two elliptic arcs. In 1990, he [40] established the well-posedness of Tricomi-Bitsadze-Lavrentjev problem for the "generalized Chaplygin equation" with discontinuous coefficient: $K(y) = \mathrm{sgn}(y)$.

In 1990, Rassias [41] suggested twelve open problems, most of which still remain unsolved, and variations of which are stated at the end of this work.

In 1992 , Rassias [43] considered the elliptic-hyperbolic and parabolic equation $Lu = K(y)u_{xx} + u_{yy} + r(x,y)u = f(x,y)$,in a simply-connected domain G bounded by an elliptic arc g_1 $(for \;\; y > 0)$: connecting points : $A = (1,0)$; $A' = (-1,0)$ and by two hyperbolic characteristic curves:

$$g_2(\equiv PA): x = 1 + \int_0^y \sqrt{-K(t)}dt \;; g_3(\equiv A'P): x = -1 - \int_0^y \sqrt{-K(t)}dt \quad (for \quad y < 0),$$

intersecting at $P \equiv P(0, y_p), \quad where \quad y_p(<0): \int_0^{y_p} \sqrt{-K(t)}dt = -1$, proved a uniqueness theorem on quasi-regular solutions in $C^2(G) \cap C(\overline{G})$ and the pertinent existence theorem on weak solutions in $L^2(G)$, and established the well-posedness of the Tricomi problem: consisting in finding a function $u = u(x, y)$ which satisfies above mixed type equation in G and boundary condition: $u = 0 \quad on \quad g_1 \cup g_2$.

In 1993 , J. M. Rassias [44] considered the 2D elliptic and hyperbolic and parabolic partial differential equation: $Lu = K(y)u_{xx} + u_{yy} + r(x, y)u = f(x, y)$, in a simply-connected domain G bounded by an elliptic arc $g_1 \quad (for \quad y > 0)$: connecting : $A = (1,0)$; $A' = (-1,0)$ and by two hyperbolic characteristic curves:

$$g_2(\equiv PA): x = 1 + \int_0^y \sqrt{-K(t)}dt \;; \; g_3'(\equiv A'P): x = -1 - \int_0^y \sqrt{-K(t)}dt \quad (for \quad y < 0),$$

intersecting at $P \equiv P(0, y_p), \quad where \quad y_p(<0): \int_0^{y_p} \sqrt{-K(t)}dt = -1$, as well as

by characteristic curves:

$$g_2' : x = \int_0^y \sqrt{-K(t)}dt \;; \; g_3 : x = -\int_0^y \sqrt{-K(t)}dt \quad (for \quad y < 0),$$

proved a uniqueness theorem on quasi-regular solutions in $C^2(G) \cap C(\overline{G})$ and the pertinent existence theorem on weak solutions in $L^2(G)$, and therefore he established the pertinent well-posedness of solutions of the Tricomi problem: consisting in finding a function $u = u(x, y)$ which satisfies above mixed type equation in G and boundary condition: $u = 0 \quad on \quad g_1 \cup g_3 \cup g_3'$.

In 1997 and 1999, Rassias ([50]-[53]) introduced the elliptic-hyperbolic and parabolic equation: $Lu = K_1(y)u_{xx} + (K_2(y)u_y)_y + r(x, y)u = f(x, y)$, in a simply-connected domain G bounded by an elliptic arc $g_1 \quad (for \quad y > 0)$: connecting : $A = (1,0)$; $A' = (-1,0)$ and by two characteristic curves:

$$g_2(\equiv PA): x = 1 + \int_0^y \sqrt{-K(t)}dt \;; g_3(\equiv A'P): x = -1 - \int_0^y \sqrt{-K(t)}dt \quad (for \quad y < 0),$$

intersecting at $P \equiv P(0, y_p), \quad where \quad y_p(<0): \int_0^{y_p} \sqrt{-K(t)}dt = -1$, proved a uniqueness theorem on quasi-regular solutions in $C^2(G) \cap C(\overline{G})$ and the pertinent existence theorem on weak solutions in $L^2(G)$, and therefore, Rassias established the well-posedness of the Tricomi problem: consisting in finding $u = u(x, y)$ which satisfies above mixed type equation in G and boundary condition: $u = 0 \quad on \quad g_1 \cup g_2$.

Techniques to prove the existence of weak solutions :

Let us consider the relative adjoint equation:

$$L^+ w = K_1(y)w_{xx} + (K_2(y)w_y)_y + r(x, y)w = f(x, y),$$

and the adjoint boundary condition: $w = 0 \quad on \quad g_1 \cup g_3$.

Then Rassias introduced the following preliminary concepts:

$$C^2(\overline{G}) = \begin{Bmatrix} u(p) \mid p=(x,y)\in \overline{G} = G\cup\partial G \ : \ u=u(p) \\ is \ \ twice-continuously \ \ differentiable \ in \ \overline{G} \end{Bmatrix};$$

$\|u\|_{C^2(\overline{G})} = \max\left\{\left|D^\alpha u(p)\right| \mid p\in\overline{G} : |\alpha|\le 2\right\}$, such that

$\alpha=(\alpha_1,\alpha_2): a_1,a_2>0, |a| = a_1+a_2$; $p=(x,y)\in\mathbb{R}^2$, and $\ \overline{p}=(\overline{x},\overline{y})\in\mathbb{R}^2$, and

$$p^\alpha = x^{\alpha_1}y^{\alpha_2}, \ \ \langle p,\overline{p}\rangle = x\overline{x}+y\overline{y}, \ \ |p| = \left(\langle p,p\rangle\right)^{1/2};$$

$$D_1=\frac{\partial}{\partial x}, \ \ D_2=\frac{\partial}{\partial y}, \ \ \left(D^\alpha u\right)(p) = (D_1^{\alpha_1}D_2^{\alpha_2}u)(p),$$

for sufficiently smooth functions ;

$$L^2(G) = \left\{u \mid \int_G |u(p)|^2\,dp<\infty\right\}, \ \ \|u\|_G = \|u\|_{L^2(G)} = \left(\int_G |u(p)|^2\,dp\right)^{1/2},$$

$with \ \ dp=dxdy$; $D(L) = \left\{u\in C^2(\overline{G}) : u=0 \ \ on \ \ g_1\cup g_2\right\}$,

the domain of the formal operator L, and $D(L^+) = \left\{w\in C^2(\overline{G}) : w=0 \ \ on \ \ g_1\cup g_3\right\}$,

the domain of the adjoint operator L^+ ; $W_2^2(G) = \left\{u \mid D^\alpha u(.)\in L^2(G), \ |\alpha|\le 2\right\}$, and

the complete normed Sobolev space with norm

$$\|u\|_2 = \|u\|_{W_2^2(G)} = (\|u\|^2_{L^2(G)} + \sum_{|\alpha|=2}\left\|D^\alpha u\right\|^2_{L^2(G)})^{1/2} = \left(\sum_{|\alpha|\le 2}\left\|D^\alpha u\right\|^2_{L^2(G)}\right)^{1/2};$$

$$W_2^2(G,bd) = \overline{D(L)_{\|\cdot\|_2}} \quad : closure \ of \ D(L) \ with \ norm \ \|.\|_2 \ ;$$

$W_2^2(G,bd^+) = \overline{D(L)_{\|\cdot\|_2}} \quad : closure \ of \ D(L^+) \ with \ norm \ \|.\|_2$, or equivalently:

$$W_2^2(G,bd^+) = \left\{w\in W_2^2(G) : \langle Lu,w\rangle_0 = \langle u,L^+w\rangle_0 \ \ for \ all \ \ u\in W_2^2(G,bd)\right\}.$$

Definition 2. 9. A function $u=u(p)\in L^2(G)$ is weak solution of Tricomi problem if $\langle f,w\rangle_0 = \langle u,L^+w\rangle_0$, holds for all $w\in W_2^2(G,bd^+)$.

Criterion 2. 10.

(i). A necessary and sufficient condition for the existence of a weak solution of Tricomi problem is that the following a-priori estimate: $\|w\|_0 \le C\left\|L^+w\right\|_0$,holds for all $w\in W_2^2(G,bd^+)$ for some positive constant C. (ii). A sufficient condition for the existence of a weak solution of Tricomi problem is that the following a-priori estimate: $\|w\|_1 \le C\left\|L^+w\right\|_0$,holds for all $w\in W_2^2(G,bd^+)$ for some positive constant C, where

$$\|w\|_0 = \|w\|_{L^2(G)} = \left(\int_G |w(p)|^2\,dp\right)^{1/2}; \ \|w\|_1 = \left(\int_G \left(|w(p)|^2 + |w_x(p)|^2 + \left|w_y(p)\right|^2\right)dp\right)^{1/2},$$

with $\|w\|_1 \ge \|w\|_0$ for all $w\in W_2^2(G,bd^+)$.

Rassias quoted that both the Hahn-Banach theorem and the Riesz representation theorem would play an important role if above criterion 1 were not employed. For the justification of the definition of weak solutions he applied Green's theorem and classical techniques in order to show that: $f = Lu \ and \ u = 0 \ on \ g_1 \cup g_2$.

Rassias proved the following theorems for the well-posedness of Tricomi problem:

Existence Theorem 2. 11. Let us consider the Tricomi problem:

$$Lu = K_1(y)u_{xx} + (K_2(y)u_y)_y + r(x,y)u = f(x,y) \quad ; \quad u = 0 \ on \ g_1 \cup g_2 .$$

Also consider the above-mentioned simply-connected domain G and conditions:

$$(R_1): r < 0 \ on \ g_2; \quad (K_1K_2)' > 0 \ on \ g_2; \quad K_i' > 0 \, (i = 1,2) \ in \ G \ ;$$

$$(R_2): (x - c_1)dy - (y + c_2)dx \geq 0: \ \text{"}star-likedness\text{"} \ on \ g_1 \ ;$$

$$(R_3): \begin{cases} 4(3r + (x-c_1)r_x + (y+c_2)r_y) + \mu_1 & \leq -4\delta_{11} < 0 \quad for \quad y \geq 0 \\ 4(2r + (x-c_1)r_x + c_2 r_y) + \mu_1 & \leq -4\delta_{12} < 0 \quad for \quad y \leq 0 \end{cases}$$

$$(R_4): \begin{cases} K_1 + (y+c_2)K_1' - \mu_2(x-c_1)^2 & \geq \delta_{21} > 0 \quad for \quad y \geq 0 \\ c_2K_1' - \mu_2(x-c_1)^2 & \geq \delta_{22} > 0 \quad for \quad y \leq 0 \end{cases} ;$$

$$(R_5): \begin{cases} K_2 + (y+c_2)K_2' - \mu_3(y+c_2)^2 & \geq \delta_{31} > 0 \quad for \quad y \geq 0 \\ 2K_2 + c_2K_2' - \mu_3(c_2)^2 & \geq \delta_{32} > 0 \quad for \quad y \leq 0 \end{cases} ;$$

$$(R_6): \int_0^y \sqrt{-K(t)}dt + c_2\sqrt{-K(y)} - c_0 < 0 \ on \ g_2 \ ,$$

where $\delta_{ij} > 0 \ (i = 1,2,3; \ j = 1,2)$; $K_i (i = 1,2); r; f \ are \ sufficiently \ smooth; c_1 = 1 + c_0$,

and $c_0; c_2; \mu_i (i = 1,2,3)$ are positive constants.

Then there exists a weak solution of the above pertinent Tricomi problem.

Uniqueness Theorem 2. 12. If one considers the mixed domain G,

the star-likedness condition:

$$(R_1): (x+1)dy - ydx \geq 0 \quad on \quad g_1 ,$$

and the coefficient conditions:

$$(R_2): \begin{cases} 2r + (x+1)r_x + yr_y < 0 & for \quad y \geq 0 \\ r + (x+1)r_x < 0 & for \quad y \leq 0 \end{cases} ; \ r < 0 \ on \ g_3 ;$$

$$(R_3): \begin{cases} K_1(y) > 0 & whenever \quad y > 0 \\ K_1(0) = 0 & \\ K_1(y) < 0 & whenever \quad y < 0 \\ K_2(y) > 0 & everywhere \quad in \quad G \end{cases} ;$$

$$(R_4): K_i'(y) > 0 \ (i = 1,2) \ in \ G ,$$

where $K_i (i = 1,2)$, and r are once-continuously differentiable and f is continuous, then this Tricomi problem (T) has at most one quasi-regular solution in G.

In 2002, Rassias [54] introduced the elliptic, bi-hyperbolic and bi-parabolic equation

$Lu = K_1(y)(M_2(x)u_x)_x + M_1(x)(K_2(y)u_y)_y + r(x,y)u = f(x,y)$,

which is parabolic on both segments: $x=0,\ 0<y\le 1\ \ ;\ \ y=0,\ 0<x\le 1$,

elliptic in the euclidean region: $G_e = \{(x,y)\in G(\subset \mathbb{R}^2): x>0,\ y>0\}$, and hyperbolic in both regions : $G_{h_1} = \{(x,y)\in G(\subset \mathbb{R}^2): x>0,\ y<0\}$; $G_{h_2} = \{(x,y)\in G(\subset \mathbb{R}^2): x<0,\ y>0\}$,

with G the simply-connected mixed domain of our pertinent Tricomi problem.

In fact, Rassias considered:

$f = f(x,y)$ continuous in G, $r = r(x,y)$ once-continuously differentiable in G, $K_i = K_i(y)\ \ (i=1,2)$ are once-continuously differentiable for $y\in[-k_1,k_2]$: $-k_1 = \inf\{y:(x,y)\in G\}$; $k_2 = \sup\{y:(x,y)\in G\}$,and $M_i = M_i(x)\ \ (i=1,2)$ are once-continuously differentiable for

$x\in[-m_1,m_2]$: $-m_1 = \inf\{x:(x,y)\in G\}$; $m_2 = \sup\{x:(x,y)\in G\}$.

Besides he assumed the following conditions:

$$K_1 = K_1(y)\begin{cases} >0 & for\quad y>0 \\ =0 & for\quad y=0 \\ <0 & for\quad y<0 \end{cases} \quad and \quad M_1 = M_1(x)\begin{cases} >0 & for\quad x>0 \\ =0 & for\quad x=0 \\ <0 & for\quad x<0 \end{cases},$$

as well as : $K_2 = K_2(y) > 0 \quad and \quad M_2 = M_2(x) > 0 \quad everywhere \quad in\ \ G$,so that

$$K = K(y) = \frac{K_1(y)}{K_2(y)}\begin{cases} >0 & for\quad y>0 \\ =0 & for\quad y=0 \\ <0 & for\quad y<0 \end{cases} \ and\ \ M = M(x) = \frac{M_1(x)}{M_2(x)}\begin{cases} >0 & for\quad x>0 \\ =0 & for\quad x=0 \\ <0 & for\quad x<0 \end{cases}.$$

Also Rassias supposed that : $\lim_{y\to 0} K(y) \quad and \quad \lim_{x\to 0} M(x) \quad exist\ \ in\ \ G$;

$$KM = K(y)M(x)\begin{cases} >0 & for\quad x>0, y>0 \\ =0 & for\quad x=0; y=0 \\ <0 & for\quad x>0, y<0; x<0, y>0 \end{cases}.$$

Rassias quoted that it is not considered here: $KM>0 \quad for \quad x<0, y<0$, and the above mixed type equation degenerates its order at the origin O(0,0). Besides Rassias considered the boundary ∂G of the domain G formed by curves: A curve g_1 which is the elliptic arc lying in the first quadrant: $x>0, y>0$ and connecting the points $A(1,0)\ \ and\ \ B(0,1)$;two characteristic arcs

$$g_2 : \int_1^x \sqrt{M(t)}dt = \int_0^y \sqrt{-K(t)}dt, \quad g_3 : \int_0^x \sqrt{M(t)}dt = -\int_0^y \sqrt{-K(t)}dt,$$

descending from the points $A(1,0)\ \ and\ \ O(0,0)$ until they terminate at a common point of intersection $P_1(x_{p_1}, y_{p_1})$ in the fourth quadrant : $x>0, y<0$; and two hyperbolic characteristic arcs $\gamma_2\ \ and\ \ \gamma_3$:

$\gamma_2 : \int_0^x \sqrt{-M(t)}dt = \int_1^y \sqrt{K(t)}dt, \quad \gamma_3 : \int_0^x \sqrt{-M(t)}dt = -\int_0^y \sqrt{K(t)}dt,$

emanating from the points $\mathrm{B}(0,1)$ *and* $O(0,0)$ until they terminate at a common point of intersection $P_2(x_{p_2}, y_{p_2})$ in the second quadrant : $x<0, y>0$. Rassias assumed the boundary condition: $u=0 \quad on \quad g_1 \cup g_2 \cup \gamma_2$, and suggested the following Problem (T): The Tricomi problem, or Problem (T) consists in finding a function $u = u(x,y)$ which satisfies the above equation in G and the boundary condition on $g_1 \cup g_2 \cup \gamma_2$ which is a "continuous" part of the boundary ∂G of G.

Definition 2. 13. A function $u = u(x,y)$ is a quasi-regular solution of Problem (T) if :

(i) $u \in C^2(G) \cap C(\overline{G}), \quad \overline{G} = G \cup \partial G;$

(ii) the Green's theorem (of the integral calculus) is applicable to the integrals

$\iint_G u_x Lu dx dy, \quad \iint_G u_y Lu dx dy$;

(iii) the boundary and region integrals, which arise, exist;

(iv) u satisfies the above mixed type equation in G and the boundary condition on $g_1 \cup g_2 \cup \gamma_2$ .Then Rassias stated and proved:

Uniqueness Theorem 2. 14. If one assumes the above conditions in the mixed domain G, and the star-likedness condition: $xdy - ydx \geq 0 \quad on \quad g_1$, and conditions:

$$\begin{cases} 2r + xr_x + yr_y < 0 \ \ for \ \ x \geq 0, y \geq 0 \\ r + xr_x < 0 \ \ for \ \ x \geq 0, y \leq 0 \\ r + yr_y < 0 \ \ for \ \ x \leq 0, y \geq 0 \end{cases} ; \ r < 0 \ \ on \ \ g_3 \cup \gamma_3 ;$$

$$\begin{cases} K_2(y) - yK_2'(y) > 0 \ for \quad y \geq 0 \\ M_2(x) - xM_2'(x) > 0 \ for \quad x \geq 0 \\ K_i'(y) > 0 \qquad in \qquad G \quad ; M_i'(x) > 0 \quad in \quad G \ , for \quad i = 1,2 \end{cases} ,$$

then this Tricomi problem (T) has at most one quasi-regular solution in G.

Case $K_1 = \mathrm{sgn}(y)$; $M_1 = \mathrm{sgn}(x)$; $K_2 = M_2 = 1$: was investigated (in1984) [32].

3. THE TRICOMI-PROTTER PROBLEM

In 2007, Rassias [55] introduced the $nD\,(n > 2)$) parabolic elliptic- hyperbolic- parabolic partial differential equation

$$Lu \equiv K_1(t)(\sum_{i=1}^{n} u_{x_i x_i}) + (K_2(t)u_t)_t + r(x,t)u = f(x,t) \tag{3.1}$$

which is parabolic on $t=0$; elliptic in $t>0$;and hyperbolic in $t<0$, for $x = (x_1, x_2, ..., x_n)$, and $K_2(0) > 0$; $K_2(t) > 0$ for all real $t \neq 0$, as well as $K_1(0) = 0$; $K_1(t) > 0$ for $t > 0$;and $K_1(t) < 0$ for $t < 0$,as well as $(\quad)_t = \partial / \partial t$.

Besides Rassias investigated the pertinent Tricomi - Protter problem of this equation. Furthermore he established uniqueness of quasi-regular solutions for the afore-mentioned Tricomi – Protter problem.

Rassias introduced the parabolic elliptic-hyperbolic equation (3.1) in a bounded simply-connected mixed domain G with a piecewise smooth boundary $\partial G = S_1 \cup S_3 \cup S_4$, where $f = f(x,t)$ is continuous in G, $r = r(x,t)$ is once-continuously differentiable in G, $K_i = K_i(t)\,(i=1,2)$ are monotone increasing continuously differentiable for $t \in [-k_1, k_2]$ with $-k_1 = \inf\{t:(x,t)\in G\}$ and $k_2 = \sup\{t:(x,t)\in G\}$, for $x = (x_1, x_2, ..., x_n)$. Let us denote $K_1(t) = K(t)K_2(t)$. Rassias quoted that the boundary ∂G of a bounded simply-connected region G of $\mathbb{R}^{n+1}(n>2)$ is formed by the following surfaces: A piecewise smooth surface S_1 : $\chi(x,t)=0$ lying in the elliptic region $G^+ = \{(x,t)\in G : t>0\}$ which intersects the parabolic plane $G_0 = \{(x,t)\in G : t=0\}$ in $|x|=1$;two characteristics S_3 , S_4 of (3.1): $S_3 : \Psi(x,t) \equiv |x| - 1 + \rho = 0$, and $S_4 : \Phi(x,t) \equiv |x| - \rho = 0$, lying in the hyperbolic region $G^- = \{(x,t)\in G : t<0\}$ and satisfying the conditions

$$K_1(t)[\sum_{i=1}^{n}(\partial\Psi/\partial x_i)^2] + K_2(t)(\partial\Psi/\partial t)^2 \,|_{S_3} = 0 \quad ; \quad K_1(t)[\sum_{i=1}^{n}(\partial\Phi/\partial x_i)^2] + K_2(t)(\partial\Phi/\partial t)^2 \,|_{S_4} = 0 \ ,$$

where $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$ and $\rho = -\int_0^t \sqrt{-K(\tau)}\, d\tau\ (>0)$ with $K(\tau) = K_1(\tau)/K_2(\tau) < 0$ for $K_1(\tau)<0$ and $K_2(\tau)>0$ if $\tau<0$. Besides S_3 ascends from a fixed point $P(0,t_p), t_p<0$ and intersects the plane G_0 in $|x|=1$ for $t<0$, and S_4 descends from the fixed origin O(0,0), and intersects S_3 for $t<0 : t>t_p$. Furthermore if G is bounded, instead of S_3 , by a piecewise smooth non-characteristic surface $S_3' : \Theta(x,t)=0$ which intersects the plane G_0 in $|x|=1$ and satisfies the non-negative condition : $K_1(t)[\sum_{i=1}^{n}(\partial\Theta/\partial x_i)^2] + K_2(t)(\partial\Theta/\partial t)^2 \,|_{S_3'} \geq 0$,then the surface S_3' lies inside the characteristic 3D-space bounded by the characteristic surfaces S_3 and S_4 of (3.1). Let us assume the boundary condition

$$u = 0 \quad \text{on} \quad S_1 \cup S_3 \ . \tag{3.2}$$

If S_3 is replaced by S_3' , then the boundary condition (3.2) is replaced accordingly. Rassias named the following problem after Protter (1977), because Protter was the pioneer investigator for a class of Goursat type problems for the wave equation in dimensions greater than 2 , fundamental for the Tricomi problem, and his results today consist a cornerstone in boundary value problems of mixed type partial differential equations.

According to Protter ([Bull. Amer.Math. Soc. ,1 (3) (1979), 534-538] ; [25]):

"very little is yet known of the nature of solutions of equations of mixed type when the number of independent variables is more than two, and that the task of studying mixed type boundary value problems in three and more dimensions appears formidable and more remote."

Rassias ([20], [23-26], [36]) generalized the Tricomi and Frankl problems in n (>2) dimensions based on Protter's original proposal to him. This generalization in multi-dimensional regions was a breakthrough for the Tricomi and Frankl problems in domains of higher than two dimensions for the generalized multi-dimensional Chaplygin equation with more general coefficients. Rassias now generalizes even further his own results obtained in his Ph. D thesis (U. C., Berkeley, 1977), through the following investigation:

The Tricomi - Protter problem, or Problem (TP) consists in finding a function $u = u(x,t)$ which satisfies the mixed type partial differential Eq. (3.1) in G and the boundary condition (3.2) on the surface portion $S_1 \cup S_3$ of the boundary ∂G of G.

If S_3 is replaced by S_3', the Tricomi - Protter problem, or Problem (TP) is recalled accordingly as the Frankl - Protter problem, or Problem (FP) .

Definition 3. 1. A function $u = u(x,t)$ is a quasi-regular solution of Problem (TP) if the following two conditions hold:

(i) the Green's theorem (of the integral calculus) is applicable to the integrals

$$\iint_G u\, Lu\, dxdt \ , \ \iint_G u_{x_i} Lu\, dxdt \ \ (i = 1,2,...,n) \ \text{and} \ \iint_G u_t\, Lu\, dxdt \ ;$$

(ii) the boundary surface and region integrals, which arise, exist; and (iii) u satisfies the mixed type Eq. (3.1) in G and the boundary condition (3.2) on $\partial G = S_1 \cup S_3 \cup S_4$.

Besides Rassias stated and proved the following general nD uniqueness theorem:

Uniqueness Theorem 3. 2. Consider the parabolic elliptic- hyperbolic Eq. (3.1) and the boundary condition (3.2), as well as the afore-described bounded simply-connected mixed domain G of the (x, t) - euclidean space $\mathbb{R}^{n+1} (n > 2)$. Besides assume that $K_2(0) > 0$; $K_2(t) > 0$ for all real $t \neq 0$, as well as $K_1(0) = 0$; $K_1(t) > 0$ for $t > 0$;and $K_1(t) < 0$ for $t < 0$.Furthermore suppose the conditions:

(R_1): The piecewise smooth surface S_1: $\chi(x,t) = 0$ is " strongly star-like",

such that $\sum_{i=1}^{n} x_i \nu_i + c(t)\nu_{n+1} |_{S_1} \geq 0$ for $t > 0$,where $\nu = (\nu_1, \nu_2, ..., \nu_n, \nu_{n+1})$ is the normal unit vector on $\partial G = S_1 \cup S_3 \cup S_4$, and

$$c(t) = \int_0^t \sqrt{K(\tau)}\, d\tau / \sqrt{K(t)} > 0 \ \text{ for } \ t > 0 \ \ , \ \ K(t) = K_1(t) / K_2(t) > 0 ,$$

with characteristic surfaces S_3 and S_4 of equation (3.1) described above ;

$$(R_2): \qquad (2a - nr) - \sum_{i=1}^{n} x_i r_{x_i} - (cr)_t + (K_2 a')' \geq 0 \ \text{ in G},$$

$$a = [(n-1)/2] + [(K_1(t)K_2(t))' c / 4K_1(t)K_2(t)]$$
$$= [(n-1)/2] + \{[[(K(t))' / K(t)] + 2[(K_2(t))' / K_2(t)]](c/4)\} ,$$

$$c = \int_0^t \sqrt{|K(\tau)|}\, d\tau / \sqrt{|K(t)|}$$

in G and $(\ )' = d(\ )/dt$;

(R_3): $\qquad R(t) = (aK_2 \sqrt{-K}\, \rho^{n-1})' + a' K_2 \sqrt{-K}\, \rho^{n-1} < 0$ for $t < 0$,

$\rho = -\int_0^t \sqrt{-K(\tau)}\, d\tau \ (> 0)$ for $t < 0$;

(R_4): $R^*(t) = K_2\,\rho^n - 2(T_{n+1}^2(t)/T'_{n+1}(t) > 0$ for $t < 0$,

with a real valued function $T_{n+1} : \mathbb{R} \to \mathbb{R}\ (n \ge 2)$, such that

$T_{n+1} = T_{n+1}(t) \to 0$ as $t \to 0^-$ and $T'_{n+1} = dT_{n+1}(t)/dt = -R(t)|_{S_4}\ (> 0)$,

and $f = f(x,t)$ is continuous in G, $r = r(x,t)$ is once-continuously differentiable in G, $K_i = K_i(t)\ (i = 1,2)$ are monotone increasing continuously differentiable for $t \in [-k_1, k_2]$ with : $-k_1 = \inf\{t : (x,t) \in G\}$, $k_2 = \sup\{t : (x,t) \in G\}$, for $x = (x_1, x_2, ..., x_n)$.

Then the above Problem (TP) has at most one quasi-regular solution in G.

Proof . We apply the well known $a,\ b_i\ (i = 1,2,...,n),\ c$ energy integral method with choices in G:

$$a = [(n-1)/2] + [(K_1(t)K_2(t))'c / 4K_1(t)K_2(t)],\ b_i = x_i,\ c = \int_0^t \sqrt{|K(\tau)|}\,d\tau / \sqrt{|K(t)|}$$

for all $i = 1,2,...,n$. Then we use the above mixed type Eq. (3.1) as well as the boundary condition (3.2). We assume two solutions u_1, u_2 of the Problem (TP). Then, we claim that $u = u_1 - u_2 = 0$ holds in the domain G. In fact, we investigate

$$0 = J = 2(lu, Lu)_0 = \iint_G 2\,lu\,Lu\,dxdt \tag{3.3}$$

where $lu = a(t)u + \sum_{i=1}^n b_i(x_i)u_{x_i} + c(t)u_t$; $a = a(t), b_i = b_i(x_i), c = c(t)$ are defined above. Thus $Lu = Lu_1 - Lu_2 = f - f = 0$ in G. We introduce the new differential identities

$$2aK_1uu_{x_ix_i} = (2aK_1uu_{x_i})_{x_i} - 2aK_1u_{x_i}^2 - ((aK_1)_{x_i}u^2)_{x_i} + (K_1a)_{x_ix_i}u^2$$
$$= (2aK_1uu_{x_i})_{x_i} - 2aK_1u_{x_i}^2;$$

$$2aK_2uu_{tt} = (2aK_2uu_t)_t - 2aK_2u_t^2 - ((aK_2)'u^2)_t + (K_2a)''u^2,$$

$$2aruu = 2aru^2;\ \ 2aK_2'uu_t = (aK_2'u^2)_t - (aK_2')'u^2;$$

$$2b_iK_1u_{x_i}u_{x_{jj}} = (2b_iK_1u_{x_i}u_{x_j})_{x_j} - (b_iK_1u_{x_j}^2)_{x_i} + (b_iK_1)_{x_i}u_{x_j}^2 - 2(b_iK_1)_{x_j}u_{x_i}u_{x_j}$$
$$= (2b_iK_1u_{x_i}u_{x_j})_{x_j} - (b_iK_1u_{x_j}^2)_{x_i} + (b_iK_1)_{x_i}u_{x_j}^2\quad (i \ne j : i, j := 1,2,...,n);$$

$$2b_iK_1u_{x_i}u_{x_ix_i} = (b_iK_1u_{x_i}^2)_{x_i} - (b_iK_1)_{x_i}u_{x_i}^2;$$

$$2b_iK_2u_{x_i}u_{tt} = (2b_iK_2u_{x_i}u_t)_t - (b_iK_2u_t^2)_{x_i} + (b_iK_2)_{x_i}u_t^2 - 2(b_iK_2)_tu_{x_i}u_t;$$

$$2b_iru_{x_i}u = (b_iru^2)_{x_i} - (b_ir)_{x_i}u^2;\ \ 2b_iK_2'u_{x_i}u_t = 2(b_iK_2)_tu_{x_i}u;$$

$$2cK_1u_{x_ix_i}u_t = (2cK_1u_{x_i}u_t)_{x_i} - (cK_1u_{x_i}^2)_t + (cK_1)'u_{x_i}^2 - 2(cK_1)_{x_i}u_{x_i}u_t$$
$$= (2cK_1u_{x_i}u_t)_{x_i} - (cK_1u_{x_i}^2)_t + (cK_1)'u_{x_i}^2\ ;\ 2cK_2u_tu_{tt} = (cK_2u_t^2)_t - (cK_2)'u_t^2;$$

$$2cru_tu = (cru^2)_t - (cr)_tu^2;\ 2cK_2'u_tu_{tt} = (cK_2'u_t^2)_t - (cK_2')'u_t^2.$$

We note that:

$$\sum_{i\neq j}^{n}(b_i\nu_j+b_j\nu_i)u_{x_i}u_{x_j}=2\sum_{i<j}^{n}(b_i\nu_j+b_j\nu_i)u_{x_i}u_{x_j}=2\sum_{i\neq j}^{n}(b_i\nu_j)u_{x_i}u_{x_j}.$$

Furthermore we employ the classical Green's theorem of the integral calculus. Therefore if $dV=dxdt$, and dS the surface element ,we get the fundamental identity

$$0=J=2\iint_G [a(t)u+\sum_{i=1}^{n}b_i(x_i)u_{x_i}+c(t)u_t]Lu\,dV$$

$$=\iint_G[2ar-\sum_{i=1}^{n}(b_ir)_{x_i}-(cr)_t+\sum_{i=1}^{n}(K_1a)_{x_ix_i}+(K_2a')']u^2dV$$

$$+\iint_G\{\sum_{i=1}^{n}[-2aK_1-(b_iK_1)_{x_i}+\sum_{j\neq i}^{n}(b_jK_1)_{x_j}+(cK_1)']u_{x_i}^2$$

$$+[-2aK_2+\sum_{i=1}^{n}(b_iK_2)_{x_i}-(cK_2)'+2cK_2']u_t^2$$

$$-2\sum_{i\neq j}^{n}(K_1b_i)_{x_j}u_{x_i}u_{x_j}-2\sum_{i=1}^{n}(K_1c)_{x_i}u_{x_i}u_t\}dV$$

$$+\oint_{\partial G}(\sum_{i=1}^{n}b_i\nu_i+c\nu_{n+1})r\,u^2dS$$

$$+\oint_{\partial G}\{2au(K_1\sum_{i=1}^{n}u_{x_i}\nu_i+K_2u_t\nu_{n+1})-u^2[\sum_{i=1}^{n}(K_1a)_{x_i}\nu_i+a'K_2\nu_{n+1}]\}dS$$

$$+\oint_{\partial G}[\sum_{j=1}^{n}(b_j\nu_j-\sum_{i\neq j}^{n}b_i\nu_i-c\nu_{n+1})K_1u_{x_j}^2+(-\sum_{i=1}^{n}b_i\nu_i+c\nu_{n+1})K_2u_t^2$$

$$+2\sum_{i<j}^{n}(b_i\nu_j+b_j\nu_i)K_1u_{x_i}u_{x_j}+2\sum_{i=1}^{n}(b_iK_2\nu_{n+1}+cK_1\nu_i)u_{x_i}u_t]dS$$

$$=I_1+I_2+J_1+J_2+J_3. \tag{3.4}$$

We note that: $A_i=-2aK_1-(b_iK_1)_{x_i}+\sum_{j\neq i}^{n}(b_jK_1)_{x_j}+(cK_1)'=0$ for $i=1,2,...,n$,

$$B=-2aK_2+\sum_{i=1}^{n}(b_iK_2)_{x_i}-(cK_2)'+2cK_2'=0.$$

Therefore $I_2=\iint_G(\sum_{i=1}^{n}A_iu_{x_i}^2+Bu_t^2)dV=0$, because $A_i=B_i=(K_1b_i)_{x_j}=(K_1c)_{x_i}=0.$

From condition (R_2) ,one gets $I_1=\iint_G[(2a-nr)-\sum_{i=1}^{n}x_ir_{x_i}-(cr)_t+(K_2a')']u^2dV\geq 0$.

We claim that $J_1=0$:

In fact, this is valid from condition (3.2) because

$\nu|_{S_4} = (\nu_1, \nu_2, ..., \nu_n, \nu_{n+1}) = -\nabla\Phi / |\nabla\Phi| \; = -(x_1, x_2, ..., x_n, N)\sqrt{-K} / N\sqrt{1-K} \,|_{S_4}$,

such that $N = \rho\sqrt{-K} = -\sqrt{-K}\,(\int_0^t \sqrt{-K(\tau)}d\tau)(>0)$ for $t<0$,and

$\nabla\Phi|_{S_4} = (\Phi_{x_1}, \Phi_{x_2}, ..., \Phi_{x_n}, \Phi_t) = (x_1/\rho, x_2/\rho, ..., x_n/\rho, \sqrt{-K}\,)$, $|\nabla\Phi| = \sqrt{1-\mathrm{K}}$,

characteristic equation: $K_1(t)[\sum_{i=1}^{n}(\partial\Phi/\partial x_i)^2] + K_2(t)(\partial\Phi/\partial t)^2 \,|_{S_4} = 0$ holds, and

$[\sum_{i=1}^{n} b_i\nu_i + c\nu_{n+1}]\,|_{S_4} = -(\sqrt{-K}/N\sqrt{1-K}\,)\{\sum_{i=1}^{n} x_i^2 + [N\int_0^t \sqrt{-K(\tau)}d\tau/\sqrt{-K(t)}]\}\,|_{S_4} = 0$.

The rest of the proof is based on analogous techniques and the application of the classical Cauchy-Schwarz-Buniakowski inequality.

We are now investigating J_2: In fact, let us note that S_4:

$x_1 = \rho\cos\theta\cos\varphi_1\cos\varphi_2...\cos\varphi_{n-3}\cos\varphi_{n-2}$; $x_2 = \rho\sin\theta\cos\varphi_1\cos\varphi_2...\cos\varphi_{n-3}\cos\varphi_{n-2}$;
$x_3 = \rho\sin\varphi_1\cos\varphi_2...\cos\varphi_{n-3}\cos\varphi_{n-2}$; $x_4 = \rho\sin\varphi_2...\cos\varphi_{n-3}\cos\varphi_{n-2}$;
…………………………………
$x_{n-2} = \rho\sin\varphi_{n-4}\cos\varphi_{n-3}\cos\varphi_{n-2}$; $x_{n-1} = \rho\sin\varphi_{n-3}\cos\varphi_{n-2}$; $x_n = \rho\sin\varphi_{n-2}$; $t = t$,

for $\theta\in[0,2\pi), \varphi_j\in[-\pi/2,\pi/2]\,(j=1,2,...,n-2); x_i\in\mathbb{R}; \mathit{and}\; t\in(-\infty,0]$.

We consider: $\overline{u}(t,\theta,\varphi_1,\varphi_2,...,\varphi_{n-2}) = u(x_1,x_2,...,x_n,t)\,|_{S_4}$

$= u(\rho\cos\theta\cos\varphi_1...\cos\varphi_{n-2}, \rho\sin\theta\cos\varphi_1...\cos\varphi_{n-2}, ..., \rho\sin\varphi_{n-2}, t)$.

Thus $du\,|_{S_4} = \sum_{i=1}^{n} u_{x_i}dx_i + u_t dt = \overline{u}_t dt + \overline{u}_\theta d\theta + \sum_{j=1}^{n-2}\overline{u}_{\varphi_j}d\varphi_j$,such that $\rho = -\int_0^t \sqrt{-K(\tau)}\;d\tau$ yielding $dt = -d\rho(t)/\sqrt{-K(t)}$. Let us denote $x_i\,|_{S_4} = f_i(t,\theta,\varphi_1,...,\varphi_{n-2})$.

Thus we obtain $dx_i\,|_{S_4} = (f_i)_t dt + (f_i)_\theta d\theta + \sum_{j=1}^{n-2}(f_j)_{\varphi_j}d\varphi_j$ and

$$\partial\overline{u}/\partial t = -[\sqrt{-K}\,(u_{x_1}\cos\theta\cos\varphi_1...\cos\varphi_{n-2} + u_{x_2}\sin\theta\cos\varphi_1...\cos\varphi_{n-2} + ...$$
$$+ u_{x_n}\sin\varphi_{n-2}) - u_t]\,|_{S_4} = \partial u/\partial T\,|_{S_4} = \partial\overline{u}/\partial T\,|_{S_4},$$

which is the derivative in the direction of the tangent vector

$T = (t_1, t_2, ..., t_n, t_{n+1}) = (\partial x_1/\partial t, \partial x_2/\partial t, ..., \partial x_n/\partial t, \partial t/\partial t)$

$= -\sqrt{-K}\,(x_1, x_2, ..., x_n, -\rho/\sqrt{-K}\,)/\rho$,

of u in the direction of one of the generators of S_4 ,such that the dot $(\bullet)$ product

$\nu\bullet T = \sum_{i=1}^{n+1}\nu_i t_i = -[\sum_{i=1}^{n}\Phi_{x_i}(\partial x_i/\partial t) + \Phi_t(\partial t/\partial t)] = 0$. Therefore,

$$K_1\sum_{i=1}^{n}u_{x_i}\nu_i+K_2u_t\nu_{n+1}\,|_{S_4}=K_2\sqrt{-K/(1-K)}[\sqrt{-K}(u_{x_1}\cos\theta\cos\varphi_1...\cos\varphi_{n-2}$$
$$+...+u_{x_n}\sin\varphi_{n-2})-u_t]=-K_2\sqrt{-K/(1-K)}\;(\partial\bar{u}/\partial t) \quad (3.5)$$

where

$\cos\theta\cos\varphi_1...\cos\varphi_{n-2}=x_1/\rho=-\sqrt{1-K}\;\nu_1,...,\sin\varphi_{n-2}=-\sqrt{1-K}\;\nu_n$.

But $-dx\,|_{S_4}=\nu_{n+1}dS=-(\Phi_t/|\nabla\Phi|)dS$. Hence

$dS\,|_{S_4}=(|\nabla\Phi|/\Phi_t]dx=(\sqrt{1-K}\;/\sqrt{-K}\;)dx=(\sqrt{1-K}\;/\sqrt{-K}\;)|J|\,d\varphi d\theta dt$,

where $dx=\prod_{i=1}^{n}dx_i,\,d\varphi=\prod_{j=1}^{n-2}d\varphi_j$; and

$J=J(t,\varphi_1,\varphi_2,...,\varphi_{n-2})=$

$\partial(x_1,x_2,...,x_{n-1},x_n)/\partial(\theta,\varphi_1,...,\varphi_{n-2},t)=\sqrt{-K}\;(-\int_0^t\sqrt{-K(\tau)}\;d\tau)^{n-1}C(\varphi)=\rho^{n-1}\sqrt{-K}\;C(\varphi)$, where

$C(\varphi)=\prod_{j=1}^{n-2}\cos^j\varphi_j=(\cos\varphi_{n-2})^{n-2}...(\cos\varphi_2)^2\cos\varphi_1\;(>0)$,

for $\varphi_j\in[-\pi/2,\pi/2]\,(j=1,2,...,n-2)$ for $n>2$ and J the Jacobian.

We note that $C(\varphi)=1,\;for\;n=1\;and\;n=2$. Therefore

$$dS\,|_{S_4}=\rho^{n-1}\sqrt{1-K}\;C(\varphi)d\varphi d\theta dt. \quad (3.6)$$

Therefore from (3.5)-(3.6), one gets

$$(K_1\sum_{i=1}^{n}u_{x_i}\nu_i+K_2u_t\nu_{n+1})dS\,|_{S_4}=-\rho^{n-1}K_2\sqrt{-K}\;C(\varphi)(\partial\bar{u}/\partial t)d\varphi d\theta dt. \quad (3.7)$$

If $F=F(x,t):\mathbb{R}^{n+1}\to\mathbb{R}$ is a given real function, we find

$$J_2^*=\int_{S_4}F(x,t)dS=\int_{S_4'}F(x,t(x))(|\nabla\Phi|/\Phi_t)dx, \quad (3.8)$$

where S_4' is the projection of S_4 into the x-space, such that

$|x|=(\sum_{i=1}^{n}x_i^2)^{1/2}$; and $|x|+\int_0^t\sqrt{-K(\tau)}d\tau=0$ (or $t=t(x)$).

Let us denote $\varphi=(\varphi_1,\varphi_2,...,\varphi_{n-2})$. Thus from (3.6) and (3.8) and denoting

$\Sigma_4=\{(\theta,\varphi,t)\in\mathbb{R}^{n+1}:0\le\theta\le\pi,-\pi/2\le\varphi_j\le\pi/2(j=1,2,...,n-2),t^*(\theta,\varphi)\le t<0\}$,

we get

$$J_2^*=\int_{\Sigma_4}F(-\int_0^t\sqrt{-K(\tau)}d\tau\cos\theta\cos\varphi_1...\cos\varphi_{n-2},...,t)|J|\sqrt{1-K}/\sqrt{-K}dtd\varphi d\theta$$
$$=\int_{\Sigma_4}F(-\int_0^t\sqrt{-K(\tau)}d\tau\cos\theta\cos\varphi_1...\cos\varphi_{n-2},...,t)\rho^{n-1}(t)C(\varphi)dtd\varphi d\theta$$

We note that Σ_4 is the region in the $\theta\varphi t - space$ into which the region S_4' is mapped under the following transformation

$$x_1 = \rho\cos\theta\cos\varphi_1\cos\varphi_2 ... \cos\varphi_{n-3}\cos\varphi_{n-2}; ...;\ x_n = \rho\sin\varphi_{n-2}. \quad (3.9)$$

Let us denote

$$\Sigma_4' = \{(\theta,\varphi,\rho) \in \mathbb{R}^{n+1} : 0 \le \theta \le \pi, -\pi/2 \le \varphi_j \le \pi/2 (j = 1,2,...,n-2), 0 \le \rho \le R(\theta,\varphi)\}$$

the region in the $\theta\varphi\rho - space$ into which either the region S_4' is mapped under the transformation (3.9) or the region Σ_4 is mapped under transformation

$$\rho = -\int_0^t \sqrt{-K(\tau)}\, d\tau. \quad (3.10)$$

In this latter case the Jacobian is

$$\partial t / \partial \rho = 1 \text{ and } d\rho / dt = -\sqrt{-K(t)}\ . \quad (3.11)$$

Thus

$$dt d\varphi d\theta = \left| dt / d\rho \right| d\rho d\varphi d\theta = (1/\sqrt{-K}) d\rho d\varphi d\theta. \quad (3.12)$$

From the geometry of S_4, we obtain

$$\nu_{n+1} dS\,|_{S_4} = (-\sqrt{-K} / \sqrt{1-K})\rho^{n-1}\sqrt{1-K}\, C(\varphi) d\theta d\varphi dt = -\sqrt{-K}\, \rho^{n-1} C(\varphi) d\theta d\varphi dt.$$

Therefore from the boundary condition (3.2), one proves

$$J_2 = -\int_{\Sigma_4} 2\alpha \bar{u} \rho^{n-1} K_2 \sqrt{-K} (\partial \bar{u} / \partial t) C(\varphi) d\theta d\varphi dt + \int_{\Sigma_4} \bar{u}^2 a' K_2 \rho^{n-1} \sqrt{-K} C(\varphi) d\theta d\varphi dt.$$

From this and integrating by parts and by virtue of the fact that $\lim_{t \to 0^-} K(t) = 0$ and that $\bar{u}$ vanishes at the upper and lower limits of S_4, we get

$$J_2 = \int_{\Sigma_4} (\bar{u})^2 [(\alpha K_2 \sqrt{-K} \rho^{n-1})' + a' K_2 \sqrt{-K} \rho^{n-1}] C(\varphi) d\theta d\varphi dt. \quad (3.13)$$

We now investigate J_3:

In fact, from condition (R_1) and from the boundary condition (3.2) we prove that

$$J_3 = \int_{S_1 \cup S_3 \cup S_4} Q(u_{x_1}, u_{x_2}, ..., u_{x_n}, u_t) dS = \int_{S_1 \cup S_3} Q(u_{x_1}, u_{x_2}, ..., u_{x_n}, u_t) dS + \int_{S_4} Q(u_{x_1}, u_{x_2}, ..., u_{x_n}, u_t) dS$$

$$= \int_{S_1 \cup S_3} (N^*)^2 [K_1 \sum_{i=1}^n \nu_i^2 + K_2 \nu_{n+1}^2][\sum_{i=1}^n x_i \nu_i + c(t)\nu_{n+1}] dS + 2\int_{S_4} (\partial \bar{u} / \partial t)^2 (K_2 / \sqrt{1-K}) \rho dS$$

$$\ge 2\int_{S_4} (\partial \bar{u} / \partial t)^2 (K_2 / \sqrt{1-K}) \rho dS$$

$$= 2\int_{\Sigma_4} (\partial \bar{u} / \partial t)^2 (K_2 \rho^n) C(\varphi) d\theta d\varphi dt \quad (3.14)$$

where N^* is the normalizing factor, such that boundary condition $u|_{S_1} = 0$ implies $0 = du|_{S_1 \cup S_3} = \sum_{i=1}^{n} u_{x_i} du_i + u_t dt; \quad u_{x_i} = N^* \nu_i \ (i = 1, 2, ..., n), u_t = N^* \nu_{n+1}$,

and $[K_1 \sum_{i=1}^{n} \nu_i^2 + K_2 \nu_{n+1}^2]|_{S_1} \geq 0$ and characteristic equation $[K_1 \sum_{i=1}^{n} \nu_i^2 + K_2 \nu_{n+1}^2]|_{S_3} = 0$,

as well as the quadratic form Q on the boundary $\partial G = S_1 \cup S_3 \cup S_4$:

$$Q = Q(u_{x_1}, ..., u_{x_n}, u_t)|_{S_1 \cup S_3 \cup S_4} = \sum_{j=1}^{n} A_j^* u_{x_j}^2 + B_t^* u_t^2 + \sum_{i<j}^{n} A_{ij}^* u_{x_i} u_{x_j} + 2\sum_{i=1}^{n} B_i^* u_{x_i} u_t$$

$$= (N^*)^2 [\sum_{i=1}^{n} x_i \nu_i + c(t)\nu_{n+1}][K_1 \sum_{i=1}^{n} \nu_i^2 + K_2 \nu_{n+1}^2]|_{S_1 \cup S_3} + 2(K_2 / \sqrt{1-K})\rho(\partial \bar{u} / \partial t)^2|_{S_4} ,$$

$$\geq (N^*)^2 [\sum_{i=1}^{n} x_i \nu_i + c(t)\nu_{n+1}][K_1 \sum_{i=1}^{n} \nu_i^2 + K_2 \nu_{n+1}^2]|_{S_1} + 2(K_2 / \sqrt{1-K})\rho(\partial \bar{u} / \partial t)^2|_{S_4}$$

$$= 2(K_2 / \sqrt{1-K})\rho(\partial \bar{u} / \partial t)^2|_{S_4} \ (\geq 0) ,$$

where on S_4 :

$$A_j^* = \sum_{j=1}^{n} (b_j \nu_j - \sum_{i \neq j}^{n} b_i \nu_i - c\nu_{n+1})K_1|_{S_4} = -2KK_2(\sqrt{-K} / N\sqrt{1-K})x_j^2 \geq 0 (j = 1, 2, ..., n),$$

$$B_t^* = (-\sum_{i=1}^{n} b_i \nu_i + c\nu_{n+1})K_2|_{S_4} = 2(K_2 \sqrt{-K} / N\sqrt{1-K})(\int_0^t \sqrt{-K(\tau)} d\tau)^2 \geq 0 ,$$

$$A_{ij}^* = \sum_{i<j}^{n} (b_i \nu_j + b_j \nu_i)K_1|_{S_4} = -2(KK_2 \sqrt{-K} / N\sqrt{1-K})x_i x_j \ (i < j; i, j := 1, 2, ..., n) ,$$

$$B_i^* = \sum_{i=1}^{n} (b_i K_2 \nu_{n+1} + cK_1 \nu_i)|_{S_4} = -2(K_2 \sqrt{-K} / \sqrt{1-K})x_i \ (i = 1, 2, ..., n) .$$

Therefore from $J_1 = 0$, we obtain

$$J_4 = J_1 + J_2 + J_3 \geq \int_{\Sigma_4} \{(\bar{u})^2 [(aK_2 \sqrt{-K}\rho^{n-1})' + a'K_2 \sqrt{-K}\rho^{n-1}] + 2(\bar{u_t})^2 (K_2 \rho^n)\} C(\varphi) d\theta d\varphi dt$$

where $\bar{u_t} = \partial \bar{u} / \partial t$.

Assuming a real valued function $T_{n+1} : \mathbb{R} \to \mathbb{R} \ (n \geq 2)$, such that $T_{n+1} = T_{n+1}(t) \to 0$ as $t \to 0^- ; T'_{n+1} = dT_{n+1}(t)/dt = -[(\alpha K_2 \sqrt{-K}\rho^{n-1})' + a'K_2 \sqrt{-K}\rho^{n-1}]|_{S_4} = -R(t)|_{S_4} \ (> 0)$ from condition (R_3) .From integration by parts and by virtue of the fact that $\lim_{t \to 0^-} K(t) = 0$ and that $\bar{u}$ vanishes at the upper and lower limits of S_4,

one gets that $\int_{\Sigma_4} T'_{n+1}(t)(\bar{u})^2 C(\varphi) d\theta d\varphi dt = -2\int_{\Sigma_4} T_{n+1}(t)\bar{u}\,\bar{u_t} C(\varphi) dt d\varphi d\theta$.

From this and Cauchy-Schwarz-Buniakowski inequality, as well as: $T'_{n+1}(t) > 0$ on S_4 $(t < 0)$ and if $T_{n+1} / \sqrt{T'_{n+1}}$ is integrable, one gets on S_4 that

$$\int_{\Sigma_4} \mathrm{T}'_{n+1}(t)(\bar{u})^2 C(\varphi) d\theta d\varphi dt = \left| -\int_{\Sigma_4} \mathrm{T}'_{n+1}(t)(\bar{u})^2 C(\varphi) d\theta d\varphi dt \right|$$

$$= 2\left| \int_{\Sigma_4} T_{n+1}(t)\bar{u}\,\overline{u_t} C(\varphi) dt d\varphi d\theta \right| = 2\left| \int [\sqrt{T'_{n+1}(t)}\,\bar{u}][(T_{n+1}(t) / \sqrt{T'_{n+1}(t)})\,\overline{u_t}\,] C(\varphi) dt d\varphi d\theta \right|$$

$$\le 2[\int_{\Sigma_4} \mathrm{T}'_{n+1}(t)(\bar{u})^2 C(\varphi)\; d\theta d\varphi dt\;]^{1/2}\;[\int_{\Sigma_4} (\mathrm{T}^2_{n+1}(t) / T'_{n+1}(t))(\overline{u_t})^2 C(\varphi) d\theta d\varphi dt]^{1/2}\;,$$

or

$$\int_{\Sigma_4} \mathrm{T}'_{n+1}(t)(\bar{u})^2 C(\varphi)\; d\theta d\varphi dt \le 4\int_{\Sigma_4} (\mathrm{T}^2_{n+1}(t) / T'_{n+1}(t))(\overline{u_t})^2 C(\varphi) d\theta d\varphi dt\,.$$

Therefore from this inequality we find :

$$J_4 = \int_{\Sigma_4} [2K_2 \rho^n (\overline{u_t})^2 - T'_{n+1}(t)(\bar{u})^2] C(\varphi) d\theta d\varphi dt$$

$$\ge 2\int_{\Sigma_4} [\mathrm{K}_2 \rho^n - 2(T^2_{n+1}(t) / T'_{n+1}(t)](\overline{u_t})^2 C(\varphi) d\theta d\varphi dt \;(> 0)\,,$$

as

$$[K_2\;\rho^n - 2(T^2_{n+1}(t) / T'_{n+1}(t)]|_{\Sigma_4} == \{[\mathrm{K}_2 \rho^n T'_{n+1}(t) - 2T^2_{n+1}(t)] / T'_{n+1}(t)\}\,|_{\Sigma_4} > 0\,,$$

or

$$T'_{n+1}(t) R^*(t) = K_2 \rho^n T'_{n+1} - 2T^2_{n+1} > 0 \text{ for } \; t < 0\,,$$

which holds by condition (R_4) ,and the proof of our above theorem is complete.

NOTE: One could easily prove the existence of weak solutions and thus the well-posedness of the Tricomi-Protter problem via well-known techniques ([29],[53]).

Remarks 3. 3.

(i). The author observed that employing a variation of the $a,\ b_i\,(i = 1, 2, ..., n),\ c$ energy integral method, one obtains the above sufficient conditions $(R_i)\,(i = 1, 2, 3)$ for the uniqueness of the quasi-regular solutions of the Tricomi - Protter boundary value problem (3.1) and (3.2).

(ii). If one takes $K_1 = K,\ K_2 = 1$ in this theorem and follows Rassias proof, one establishes the uniqueness result of Rassias Ph. D. dissertation (1977) [20].

(iii). The following case :

$K_2(0) = 0$; $K_2(t) > 0$ for all real $t \ne 0$, as well as $K_1(0) = 0$; $K_1(t) > 0$ for $t > 0$;and $K_1(t) < 0$ for $t < 0$, where the order of the mixed type equation (3.1) is degenerated at $t = 0$, is analogous in investigating pertinent quasi-regular solutions of (3.1).

In 2007, J. M. Rassias and A. Hasanov [56] considered the elliptic type equation with singular coefficient:

$$L_{\alpha}^{\lambda}(u) \equiv u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x} u_x + \lambda^2 u = 0,\ 0 < 2\alpha < 1,\ \lambda = \lambda_1 + i\lambda_2, \lambda_1, \lambda_2 \in \mathbb{R},$$

in the domain $D = \{(x, y, z): 0 < x, -\infty < y < +\infty, -\infty < z < +\infty\}$.

Then they found fundamental solutions of the above equation , expressed through confluent hypergeometric functions of Kummer $H_3(a, b; c; x, y)$ from two arguments. Also they proved by means of expansion confluent hypergeometric functions of Kummer, that the constructed solutions have a singularity of the order $1/r$ at $r \to 0$ .Besides, if : $\lambda^2 = -\mu^2$,they solved some boundary value problems in domain D.

In 2007, A. Hasanov [10] considered *the generalized Rassias bi-hyperbolic equation:*

$$R(u) \equiv y^m z^k u_{xx} - x^n z^k u_{yy} + x^n y^m u_{zz} = 0; \quad m, n, k := \text{costant } (> 0),$$

in the field of : $\mathbb{R}_+^3 = \{(x, y, z): x > 0, y > 0, z > 0\}$.

By means of a suitable change of variables he reduced this generalized Rassias equation to a system of hypergeometric equations for the function of Lauricella of three variables and found eight linearly independent particular solutions of the said system of hypergeometric equations. Besides he found properties of these particular solutions by virtue of decomposition of the hypergeometric function of Lauricella.

In 2007, J. M. Rassias and G. Wen [57] focused their investigation on the oblique derivative problem for general second order mixed type equations with a nonsmooth parabolic degenerate line, which included the famous Tricomi problem as a special case. First, they gave the formulation of the above problem, and then proved the solvability of the pertinent problem for the mixed equations with nonsmooth degenerate line.

Besides they introduced a new notation, such that the second order equation of mixed type could be reduced to the mixed complex equation of first order and then they used the advantage of the complex analytic method ; otherwise, this method could not be employed.

In 2007, G. Wen, D. Chen and X. Cheng [65] investigated:

The General Tricomi-Rassias (or GTR) problem for the generalized Chaplygin equation. This is one general oblique derivative problem that includes: *The exterior Tricomi-Rassias Problem* as a special case. The investigation of these results are established by the employment of a new method. In 2007, G. Wen [66] has recently written an excellent book in mixed type equations.

Additional references are the following: ([3],[6],[11-13],[15],[18],[29-30],[34-35],[38],[42],[45-49],[51-52],[58],[61]).

4. Open Problems on Mixed Type Equations and Systems

Let us suggest the following twelve open problems:

Problem 4. 1. An open question concerns the regularity of solutions for the boundary value problems of mixed type in the sense of Tricomi and Frankl.

Problem 4. 2. An open question remains of proving the well-posedness of regular solutions without restrictions on the functional coefficients or the geometrical size or the shape of the mixed domain.

Problem 4. 3. No serious work has been established on nonlinear initial and boundary value problems of mixed type in three and more dimensions.

Problem 4. 4. The problem concerning the solution of elliptic systems in a domain on the boundary of which the type degenerates is not sufficiently investigated.

Problem 4. 5. Little is known about the Cauchy problem for hyperbolic equations of order higher than two with boundary conditions on the curve of parabolic degeneracy.

Problem 4. 6. The correct statement of the problem for equations of mixed type in multi-dimensional regions still remains to be a very intricate mathematical problem.

Problem 4. 7. The study of higher order partial differential equations and systems of partial differential equations of mixed type requires much more investigation.

Problem 4. 8. It would be very interesting to clear up the question whether there is an extremal principle for the initial and boundary value problems of mixed type.

Problem 4. 9. One of the most important problems of mathematical physics is the

investigation of the properties of solutions of partial differential equations of mixed type with boundary conditions in the sense of Tricomi and Frankl.

Problem 4. 10. To consider regions of mixed type multi-connected with parabolic lines of degeneracy replaced by arbitrary curves (not necessarily straight lines) is a question of a very high demand and difficult to handle.

Problem 4. 11. The existence problem for regular transonic flow around given general profile with given velocity at ∞ is very difficult to be solved completely.

Problem 4. 12. To solve the Tricomi, Frankl and Bitsadze-Lavrent'ev problems for any one of the following eight pertinent mixed type partial differential equations:

$$(y-x)u_{xx}+u_{yy}+r(x,y)u=f(x,y);\quad (x^2+y^2-1)u_{xx}+u_{yy}+r(x,y)u=f(x,y);$$

$$(y-x^2)u_{xx}+u_{yy}+r(x,y)u=f(x,y);\quad (x^2-y^2-1)u_{xx}+u_{yy}+r(x,y)u=f(x,y);$$

$$z(u_{xx}+u_{yy})+xu_{zz}+r(x,y,z)u=f(x,y,z);\ z(u_{xx}-u_{yy})+xu_{zz}+r(x,y,z)u=f(x,y,z);$$

$$(z-y)u_{xx}+(z-x)u_{yy}+u_{zz}+r(x,y,z)u=f(x,y,z);$$

$$\operatorname{sgn}(z-y)u_{xx}+\operatorname{sgn}(z-x)u_{yy}+u_{zz}+r(x,y,z)u=f(x,y,z).$$

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REFERENCES

[1]. A. Busemann , "The non- existence of transonic flows", Proceedings of Symposia in Applied Mathematics, **4**(1953), 29-39.

[2]. S. A. Chaplygin , "On gas jets", Sci. Ann. Imper. Univ. Moscow, Publ. 21, 1904). Translation: Brown Univ., 1944.

[3]. D. Chen and G. Wen, "Initial-oblique derivative problem for nonlinear parabolic equations in high dimensional domains",

Euler's Tri-centenn. Birthday Anniversary Vol. FIDA; in:" Mixed Type Partial Differential Equations (MT. PDE)", Intern. J. Appl. Math. Stat., **8** (M07) (2007), 8-19.

[4]. A. Douglis , "The problem of Cauchy for linear hyperbolic equations of 2nd order", Comm. Pure Appl. Math. **7** (1954), 271-295.

[5]. F. I. Frankl , "Problems of Chaplygin for mixed subsonic and supersonic flows", Izv.Akad. Nauk SSSR ser. Mat. **9**(2)(1945), 121-143.

[6]. K. O. Friedrichs, "Symmetric positive linear differential equations",

Comm. Pure and Appl. Math. ,**11** (1958), 333-418.

[7]. L. Garding, "Le Probleme de Goursat pour l' Equation des Ondes", C. R. du Onzieme Congres de Math. Scand. Trondheim ,1949.

[8]. S. Gellerstedt, "Sur un Probleme aux Limites pour une Equation Lineaire aux Derivees Partielles du Second Ordre de Type Mixte", Doctoral Thesis, Uppsala,1935; Jbuch Fortschritte Math. 61, 1259.

[9]. G. Guderley, "On the presence of shocks in mixed subsonic-supersonic flow patterns", Advanced in Applied Mechanics, Academic Press, New York, **3**(1953), 145-184.

[10]. A. Hasanov, "Some solutions of generalized Rassias's equation",

Euler's Tri-centenn. Birthday Anniv. Vol. FIDA; in:" Mixed Type Partial Differential Equations (MT. PDE)", Intern. J. Appl. Math. Stat. , **8** (M07) (2007), 20-29.

[11]. A. Hasanov, "The solution of the Cauchy problem for generalized Euler-Poisson-Darboux equation",

Euler's Tri-centenn. Birthd. Anniv. Vol. FIDA; in:" Mixed Type Partial Differential Equations (MT. PDE)", Intern. J. Appl. Math. Stat., **8** (M07) (2007), 30-43.

[12]. E. T. Karimov, "Uniqueness of the solutions of one non-local boundary value problem for the mixed parabolic type equation",

Euler's Tri-centenn. Birthd. Anniv. Vol. FIDA; in:" Mixed Type Partial Differential Equations (MT. PDE)", Intern. J. Appl. Math. Stat., **8** (M07) (2007), 44-48.

[13]. A. G. Kuz'min, "Non-uniqueness of transonic flow past a flattened airfoil",

Euler's Tri-centenn. Birthd. Anniv. Vol. FIDA; in:" Mixed Type Partial Differential Equations (MT. PDE)", Intern. J. Appl. Math. Stat., **8** (M07) (2007), 49-57.

[14]. M. A. Lavrent'ev and A. V. Bitsadze, "On the problem of equations of mixed type", Dokl. Akad. Nauk SSSR, **70**(3)(1950), 373-376.

[15]. K. R. Payne, "Multiplier methods for mixed type equations",

Euler's Tri-centenn. Birthd. Anniv. Vol. FIDA; in:" Mixed Type Partial Differential Equations (MT. PDE)", Intern. J. Appl. Math. Stat., **8** (M07) (2007), 58-75.

[16]. M. H. Protter, "Uniqueness Theorems for the Tricomi problem", J. Rat. Mechanics and Analysis, **2**(1)(1953),107-114.

[17]. M. H. Protter , "New boundary value problems for the wave equation and equations of mixed type", J. Rat. Mechanics and Analysis, **3**(4)(1954),435-446.

[18]. M. H. Protter, "An existence theorem for the generalized Tricomi problem", Duke Math. J. , **21** (1954), 1-7.

[19]. M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations , Prentice-Hall, Englewood Cliffs, N. J., USA, 1967.

[20]. J. M. Rassias, "Mixed type partial differential equations in $n > 2$ dimensions", Ph. D. Dissertation, U. C. Berkeley, (1977), 1-135.

[21]. J. M. Rassias, "The bi-hyperbolic degenerate boundary value problem in $\mathbb{R}^3$", Eleutheria, (2) (1979), 468-473.

[22]. J. M. Rassias, "A new bi-hyperbolic boundary value problem in the Euclidean space", Bull. Acad. Polon. Sci. , **28** (11-12) (1980), 443-446.

[23]. J. M. Rassias , "On a defective theorem on elliptic-hyperbolic equations", Bull. Soc. Roy. Sci. Liege, **49** (9-10) (1980), 307-309.

[24]. J. M. Rassias , " Mixed type partial differential equations in $\mathbb{R}^n$ ", Tamkang J. Math. ,**12** (2)(1981), 177-181.

[25]. J. M. Rassias , "A uniqueness theorem for the generalized Frankl-Tricomi problem", Bull. Sci. Math. ,**105** (1981), 321-327.

[26]. J. M. Rassias , “A new mixed type boundary value problem”,Bull. Sci. Math. ,**105** (1981), 329-336.

[27]. J. M. Rassias, “The bi-hyperbolic degenerate boundary value problem in $\mathbb{R}^3$”, Discuss. Math. **5** (1982), 101-104.

[28]. J. M. Rassias, “A maximum principle in $\mathbb{R}^{n+1}$”, J. Math. Anal. Appl. **85** (1) (1982), 106-113.

[29]. J. M. Rassias, “Weak solutions of the FRANKL problem in the 4-dimensional Euclidean space”, Bull. Acad. Polon. Sci. ,**30** (3-4) (1982), 476-483.

[30]. J. M. Rassias, “An application of the theory of positive symmetric systems to a degenerate multidimensional hyperbolic equation in $\mathbb{R}^3$”, SERDICA Bulg. Math. Publ., **8** (1982), 235-242.

[31]. J. M. Rassias, “On the Tricomi problem with two parabolic lines of degeneracy”, Bull. Inst. Math. Acad. Sinica, **12** (1)(1983), 51-56.

[32]. J. M. Rassias, “ The extended Bitsadze-Lavrent’ev-Tricomi boundary value problem”,Rend. Circ. Mat. Palermo, **33** (1984), 255-264.

[33]. J. M. Rassias, “Extended Bitsadze-Lavrent’ev problem with two parabolic lines of degeneracy and two elliptic arcs in euclidean plane”, Comp. Rend. Acad. Bulg. Sci., **38** (1) (1985), 1-4 .

[34]. J. M. Rassias, Mathematical Analysis, BSB Teubner, Leipzig, **79** (1985).

[35]. J. M. Rassias, Mixed Type Equations, BSB Teubner, Leipzig, **90** (1986).

[36]. J. M. Rassias , “On the well-posedness of the extended Chaplygin problem in a multidimensional region”, Comp. Rend. Acad. Bulg. Sci., **41** (2) (1988).

[37]. J. M. Rassias , “On three new generalized uniqueness theorems of the Tricomi problem for non-linear mixed type equations”, Jour. Math. Phys. Sci., **22** (6)(1988), 681-695 .

[38]. J. M. Rassias, Lecture Notes on Mixed Type Partial Differential Equations, World Sci.,1990.

[39]. J. M. Rassias, “On the exterior Tricomi and Frankl problem” in:

Lecture Notes on Mixed Type Partial Differential Equations, World Sci.,1990,73-85.

[40]. J. M. Rassias, “The well-posed Tricomi-Bitsadze-Lavrentjev problem in the Euclidean plane”, Atti Accad. Scienze di Torino, **124** (3-4) (1990), 73-83.

[41]. J. M. Rassias, **“**Open Problems**”** in:

Lecture Notes on Mixed Type Partial Differential Equations, World Sci.,1990.

[42]. J. M. Rassias, Counter Examples in Differential Equations and Related Topics, World Sci.,1991.

[43]. J. M. Rassias, “On the well-posed Tricomi problem in $\mathbb{R}^2$”, Discuss. Math. ,**12** (1992), 85-93.

[44]. J. M. Rassias, “The well-posed Tricomi problem of two kinds”, J. Math. Phys. Sci., **27** (6) (1993), 383-393.

[45]. J. M. Rassias, Functional Analysis, Approximation Theory and Numerical Analysis, World Sci. Publ. USA, 1994.

[46]. J. M. Rassias, Geometry, Analysis and Mechanics, World Sci. Publ. USA, 1994.

[47]. J. M. Rassias, “The well-posed Tricomi problem in the Euclidean plane”, in: Geometry analysis and Mechanics, World Sci. Publ., 1994, 189-195.

[48]. J. M. Rassias, Mathematical Equations and Inequalities, **I** (1997)**,** GREECE.

[49]. J. M. Rassias, “Bitsadze-Lavrent’ev problem”, Encyclopaedia of Mathematics, Kluwer Academic Publ., 1997, 526-529.

[50]. J. M. Rassias , “Uniqueness of quasi-regular solutions for a parabolic elliptic- hyperbolic Tricomi problem”, Bull. Inst. Math. Acad. Sin.,**25** (4) (1997), 277-287.

[51]. J. M. Rassias, Mathematical Equations and Inequalities, **II** (1999)**,** GREECE.

[52]. J. M. Rassias, Advances in Equations and Inequalities, Hadronic Press,1999.

[53]. J. M. Rassias , “Existence of weak solutions for a parabolic elliptic- hyperbolic Tricomi problem”, Tsukuba J. Math., **23** (1) (1999), 37-54.

[54]. J. M. Rassias, “Uniqueness of quasi-regular solutions for a bi-parabolic elliptic bi-hyperbolic Tricomi problem”, Complex Variables,**47** (8) (2002), 707-718 .

[55]. J. M. Rassias, “Tricomi - Protter problem of nD mixed type equations”,

Euler’s Tri-centenn. Birthd. Anniv. Vol. FIDA; in:” Mixed Type Partial Differential Equations (MT. PDE)”, Intern. J. Appl. Math. Stat., **8** (M07) (2007), 76-86.

[56]. J. M. Rassias and A. Hasanov, “Fundamental solutions of two degenerated elliptic equations and solutions of boundary value problems in infinite area”,

Euler’s Tri-centenn. Birthd. Anniv. Vol. FIDA; in:” Mixed Type Partial Differential Equations (MT. PDE)”, Intern. J. Appl. Math. Stat., **8** (M07) (2007), 87-95.

[57]. J. M. Rassias and G. Wen, “Solvability of the oblique derivative problem for second order equations of mixed type with nonsmooth degenerate curve”,

Euler’s Tri-centenn. Birthd. Anniv. Vol. FIDA; in: ” Mixed Type Partial Differential Equations (MT. PDE)”, Intern. J. Appl. Math. Stat., **8** (M07) (2007), 96-111.

[58]. M. S. Salakhitdinov and B. Islomov, “Boundary value problems for an equation of mixed type with two interior lines of degeneracy”, Soviet Math. Dokl., **43** (1991), 235-238.

[59]. D. Sather, “Maximum properties of Cauchy’s problem in 3-dimensional space-time”, Arch. Rational Mech. Anal., **18** (1965), 14-26 .

[60]. D. Sather, “A maximum property of Cauchy’s problem in n-dimensional space-time, Arch. Rational Mech. Anal. ,**18** (1965), 27-38.

[61]. M. M. Smirnov, Equations of Mixed Type, AMS ,Providence, RI, 1978.

[62]. S. Soboleff, “Quelques Problemes Limites Nouveaux pou les Equations aux Derivees Partielles du Type Hyperbolique”, Mat. Sbornik, **11**(1942), 155-203.

[63]. F. G. Tricomi, “Sulle Equazioni Lineari alle derivate Parziali di 2^0 Ordine, di Tipo Misto”, Atti Accad. Naz. dei Lincei, **14** (5) (1923), 133-247.

[64]. H. F. Weinberger, “A maximum property of Cauchy’s problem in 3-dimensional space-time”, Proc. Amer. Math. Soc. (1961), 91-99.

[65]. G. Wen, D. Chen and X. Cheng, “General Tricomi-Rassias problem and oblique derivative problem for generalized Chaplygin equations”, J. Math. Anal. Appl. (2007), doi: 10.1016/jmaa.2006.11.041 .

[66]. G. Wen, Elliptic, Hyperbolic and Mixed Complex Equations with Parabolic Degeneracy including Tricomi-Bers and Tricomi-Frankl-Rassias Problems, to appear,2007.