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ULAM STABILITY FOR THE ORTHOGONALLY GENERAL EULER - LAGRANGE TYPE FUNCTIONAL EQUATION

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ABSTRACT

In this paper, J. M. Rassias introduces the general Euler - Lagrange type functional equation of the form

$$f(mx+y)+f(mx-y)=2f(x+y)+2f(x-y)+2(m^2-2)f(x)-2f(y) \qquad (*)$$

for any arbitrary but fixed real constant m *with* $m\neq 0; m\neq \pm 1; m\neq \pm\sqrt{2}$ *. We investigate the Ulam stability for the orthogonally general Euler - Lagrange type functional equation (*) controlled by the mixed type product-sum function*

$$(x,y)\rightarrow \epsilon\left[\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| y \|_E^{2p}\right)\right]$$

introduced by the third author of this paper, and by a non-negative function with $x\perp y$.

Keywords: Hyers - Ulam - Rassias stability , Ulam - Gavurta - Rassias stability, Orthogonally Euler -Lagrange functional equation, Orthogonality space, Quadratic mapping.

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1 Introduction

In 1940, S. M. Ulam [27] raised the question concerning the stability of group homomorphisms:

Let G_1 *be a group and let* G_2 *be a metric group with the metric* $\rho(.,.)$. *Given* $\epsilon > 0$, *does there exists a* $\delta > 0$ *such that if a function* $h : G_1 \rightarrow G_2$ *satisfies the inequality* $\rho(h(xy), h(x)h(y)) < \delta$ *for all* $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \rightarrow G_2$ *with* $\rho(h(x), H(x)) < \epsilon$ *for all* $x \in G_1$*?*

D. H. Hyers [6] answered this problem under the assumption that the groups are Banach spaces. Th. M. Rassias [24] generalized the theorem of Hyers for approximately linear mappings. The stability phenomenon that was proved by Th. M. Rassias [24] is called the Hyers - Ulam - Rassias Stability.

J. M. Rassias [11-23] solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings. In 2005, J. M. Rassias [23] solved Euler- Lagrange type quadratic functional equation of the form

$$Q(m_1a_1x_1 + m_2a_2x_2) + m_1m_2Q(a_1x_1 - a_2x_2) = (m_1a_1^2 + m_2a_2^2)(m_1Q(x_1) + m_2Q(x_2))$$

and discussed its Ulam stability problem.

The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), x \perp y \tag{1.1}$$

in which $\perp$ is an abstract orthogonality symbol, was investigated by S. Gudder and D. Strawther [5]. R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.1) in [4].

We now introduce the concepts of orthogonality vector space, orthogonality space and orthogonality normed space and then proceed to prove our main results.

Definition 1.1. A vector space X is called an *orthogonality vector space* if there is a relation $x \perp y$ on X such that

(i) $x \perp 0,\ \ 0 \perp x$ for all $x \in X$;

(ii) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(iii) $x \perp y$, $ax \perp by$ for all $a, b \in \mathbb{R}$;

(iv) if P is a two-dimensional subspace of X ; then

(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;

(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.

Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector $x,\ y$ define $x \perp y$ iff $x,\ y$ are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x,\ y \in X$. The pair $(x, \perp)$ is called an *orthogonality space*. It becomes *orthogonality normed space* when the orthogonality space is equipped with a norm.

Definition 1.2. Let X be an orthogonality space and Y be a real Banach space. A mapping $f : X \to Y$ is called *orthogonally quadratic* if it satisfies the so called orthogonally Euler-Lagrange (or Jordan - von Neumann) quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.2}$$

for all $x, y \in X$ with $x \perp y$, (see [15]).

The orthogonality Hilbert space for the orthogonally quadratic functional equation (1.2) was first investigated by F. Vajzovic [28] . Recently Ulam - Gavruta - Rassias stability for the orthogonally Euler - Lagrange type functional equation of the form

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 4f(x) - 2f(y) \tag{1.3}$$

was investigated by Ravi and Arunkumar [26].

In this paper, we investigate the Ulam stability for the orthogonally general Euler - Lagrange type functional equation

$$f(mx+y)+f(mx-y)=2f(x+y)+2f(x-y)+2(m^2-2)f(x)-2f(y) \tag{1.4}$$

for all $x, y \in X$ with $x \perp y$, controlled by the mixed type product-sum function

$$(x,y) \to \epsilon \left\{ \| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| y \|_E^{2p} \right) \right\},$$

a concept introduced by the third author of this paper, and by a general non-negative function. Note that the general Euler - Lagrange type functional equation (1.4) is equivalent to the standard Euler - Lagrange equation (1.2).

A mapping $f : X \to Y$ is called orthogonally quadratic if it satisfies the quadratic functional equation (1.4) for all $x, y \in X$ with $x \perp y$ where X be an orthogonality space and Y be a real Banach space.

2 Stability of the Functional Equation (1.4)

In this section, let $(E, \perp)$ denote an orthogonality normed space with norm $\| \cdot \|_E$ and $(F, \| \cdot \|_F)$ is a Banach space.

Theorem 2.1. *Let $f : E \to F$ be a mapping which satisfying the inequality*

$$\begin{aligned} &\| f(mx+y)+f(mx-y)-2f(x+y)-2f(x-y)-2(m^2-2)f(x)+2f(y) \|_F \\ &\qquad \le \epsilon \left\{ \| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| x \|_E^{2p} \right) \right\} \end{aligned} \tag{2.1}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon, p > 0$ and either

$$m > 1; p < 1 \ or \ m < 1; p > 1 \ with \ m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2} \ \ and -1 \neq |m|^{p-1} < 1.$$

Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}} \tag{2.2}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler - Lagrange quadratic mapping such that

$$\| f(x) - Q(x) \|_F \le \frac{\epsilon}{2|m^2 - m^{2p}|} \|x\|_E^{2p} \tag{2.3}$$

for all $x \in E$.

Proof. Replacing (x, y) with $(0, 0)$ in (2.1), we obtain $2|2-m^2| \, \| f(0) \| = 0$ or $f(0) = 0$ if $m^2 \neq 2$. Again substituting (x, y) by $(x, 0)$ in (2.1), we get

$$\left\| f(mx) - m^2 f(x) \right\|_F \le \frac{1}{2} \epsilon \, \|x\|_E^{2p}$$

$$(i.e)., \qquad \left\|\frac{f(mx)}{m^2} - f(x)\right\|_F \leq \frac{1}{2}\frac{\epsilon}{m^2}\|x\|_E^{2p} \, (m \neq 0) \tag{2.4}$$

for all $x \in E$. Now replacing x by mx and dividing by m^2 in (2.4) and then adding the resulting inequality with (2.4), we obtain

$$\left\|\frac{f(m^2x)}{m^4} - f(x)\right\|_F \leq \frac{1}{2}\frac{\epsilon}{m^2}\left(1 + \frac{m^{2p}}{m^2}\right)\|x\|_E^{2p} \tag{2.5}$$

for all $x \in E$. Using induction on n we obtain that

$$\begin{aligned}\left\|\frac{f(m^nx)}{m^{2n}} - f(x)\right\|_F &\leq \frac{1}{2}\frac{\epsilon}{m^2}\sum_{k=0}^{n-1}\frac{m^{2pk}}{m^{2k}}\|x\|_E^{2p} \\ &\leq \frac{1}{2}\frac{\epsilon}{m^2}\sum_{k=0}^{\infty}\frac{m^{2pk}}{m^{2k}}\|x\|_E^{2p}\end{aligned} \tag{2.6}$$

for all $x \in E$. In order to prove the convergence of the sequence $\{f(m^nx)/m^{2n}\}$ replace x by m^lx and divide by m^{2l} in (2.6), for any $n, l > 0$, we obtain

$$\begin{aligned}\left\|\frac{f(m^{n+l}x)}{m^{2(l+n)}} - \frac{f(m^lx)}{m^{2l}}\right\|_F &= \frac{1}{m^{2l}}\left\|\frac{f(m^{n+l}x)}{m^{2n}} - f(m^lx)\right\|_F \\ &\leq \frac{1}{2}\frac{\epsilon}{m^2}\frac{1}{m^{2l(1-p)}}\sum_{k=0}^{\infty}\frac{m^{2pk}}{m^{2k}}\|x\|_E^{2p}.\end{aligned} \tag{2.7}$$

Since $m^{2(1-p)} < 1$, the R.H.S of (2.7) tends to 0 as $l \to \infty$ for all $x \in E$. Thus $\{f(m^nx)/m^{2n}\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ such that

$$Q(x) = \lim_{n\to\infty}\frac{f(m^nx)}{m^{2n}} \quad \forall x \in E.$$

By letting $n \to \infty$ in (2.6), we arrive the formula (2.3) for all $x \in E$. To prove Q satisfies (1.4), replace $(x,\ y)$ by $(m^nx,\ m^ny)$ in (2.1) and divide by m^{2n} then it follows that

$$\begin{aligned}&\frac{1}{m^{2n}} \parallel f(m^n(mx+y)) + f(m^n(mx-y)) - 2f(m^n(x+y)) - 2f(m^n(x-y)) \\ &-2(m^2-2)f(m^nx) + 2f(m^ny) \parallel_F \leq \frac{\epsilon}{m^{2n}}\left\{\parallel m^nx \parallel_E^p \parallel m^ny \parallel_E^p + \left(\parallel m^nx \parallel_E^{2p} + \parallel m^ny \parallel_E^{2p}\right)\right\}.\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\parallel Q(mx+y) + Q(mx-y) - 2Q(x+y) - 2Q(x-y) - 2(m^2-2)Q(x) + 2Q(y) \parallel_F \leq 0.$$

which gives

$$Q(mx+y) + Q(mx-y) = 2Q(x+y) + 2Q(x-y) + 2(m^2-2)Q(x) - 2Q(y)$$

for all $x, y \in E$ with $x \perp y$. Therefore $Q : E \to F$ is an orthogonally Euler - Lagrange quadratic mapping which satisfies (1.4). To prove the uniqueness of Q, let Q' be another orthogonally Euler - Lagrange quadratic mapping satisfying (1.4) and the inequality (2.3). We have

$$\begin{aligned}\left\|Q(x) - Q'(x)\right\|_F = & \frac{1}{m^{2n}}\left\{\|Q(m^n x) - f(m^n x)\|_F + \left\|f(m^n x) - Q'(m^n x)\right\|_F\right\} \\ \leq & \frac{1}{2}\frac{2\,\epsilon}{m^2}\sum_{j=0}^{\infty}\frac{1}{m^{2(k+n)(1-p)}}\|x\|_E^{2p} \\ & \to 0 \quad \text{as} \quad n \to \infty\end{aligned}$$

for all $x \in E$. Therefore Q is unique. This completes the proof of the theorem. □

Theorem 2.2. *Let $f : E \to F$ be a mapping which satisfying the inequality*

$$\begin{aligned}&\| f(mx + y) + f(mx - y) - 2f(x + y) - 2f(x - y) - 2(m^2 - 2)f(x) + 2f(y) \|_F \\ &\qquad \leq \epsilon\left\{\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| x \|_E^{2p}\right)\right\}\end{aligned} \tag{2.8}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon, p > 0$ and either

$$m > 1; p > 1 \; or \; m < 1; p < 1 \; with \; m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2} \quad and -1 \neq |m|^{1-p} < 1.$$

Then the limit

$$Q(x) = \lim_{n\to\infty} m^{2n} f\left(\frac{x}{m^n}\right) \tag{2.9}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler - Lagrange quadratic mapping such that

$$\| f(x) - Q(x) \|_F \leq \frac{\epsilon}{2|m^{2p} - m^2|}\|x\|_E^{2p} \tag{2.10}$$

for all $x \in E$.

Proof. Replacing x by $\frac{x}{m}(m \neq 0)$ in (2.4), we get

$$\left\|f(x) - m^2 f\left(\frac{x}{m}\right)\right\|_F \leq \frac{1}{2}\frac{\epsilon}{m^{2p}}\|x\|_E^{2p}\,(m \neq 0) \tag{2.11}$$

for all $x \in E$. Now replacing x by $\frac{x}{m}$ and multiply by m^2 in (2.11) and summing the resultant inequality with (2.11), we arrive

$$\left\|f(x) - m^4 f\left(\frac{x}{m^2}\right)\right\|_F \leq \frac{1}{2}\frac{\epsilon}{m^{2p}}\left(1 + \frac{m^2}{m^{2p}}\right)\|x\|_E^{2p} \tag{2.12}$$

for all $x \in E$. Using induction on n we obtain that

$$\begin{aligned}\left\|f(x) - m^{2n} f\left(\frac{x}{m^n}\right)\right\|_F &\leq \frac{1}{2}\frac{\epsilon}{m^{2p}}\sum_{k=0}^{n-1}\frac{m^{2k}}{m^{2pk}}\|x\|_E^{2p} \\ &\leq \frac{1}{2}\frac{\epsilon}{m^{2p}}\sum_{k=0}^{\infty}\frac{m^{2k}}{m^{2pk}}\|x\|_E^{2p}\end{aligned} \tag{2.13}$$

for all $x \in E$. In order to prove the convergence of the sequence $\{m^{2n} f\left(\frac{x}{m^n}\right)\}$, replace x by $\frac{x}{m^l}$ and multiply by m^{2l} in (2.13), for any $n, l > 0$,we obtain

$$\left\| m^{2(n+l)} f\left(\frac{x}{m^{l+n}}\right) - m^{2l} f\left(\frac{x}{m^l}\right)\right\|_F = m^{2l} \left\| m^{2n} f\left(\frac{x}{m^{l+n}}\right) - f\left(\frac{x}{m^l}\right)\right\|_F$$
$$\leq \frac{1}{2}\,\frac{\epsilon}{m^{2p}}\,\frac{1}{m^{2l(p-1)}} \sum_{k=0}^{\infty} \frac{m^{2k}}{m^{2pk}} \|x\|_E^{2p}. \qquad (2.14)$$

Since $m^{2(p-1)} < 1$, the R.H.S of (2.14) tends to 0 as $l \to \infty$ for all $x \in E$. Thus $\{m^{2n} f\left(\frac{x}{m^n}\right)\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ such that

$$Q(x) = \lim_{n\to\infty} m^{2n} f\left(\frac{x}{m^n}\right) \qquad \forall\, x \in E.$$

By letting $n \to \infty$ in (2.13), we arrive the formula (2.10) for all $x \in E$. To show that Q is unique and it satisfies (1.4), the proof is similar to that of Theorem 2.1 □

Theorem 2.3. *Let E be a real orthogonality normed linear space and F be a real complete normed linear space. Assume in addition that $f : E \to F$ is an approximately quadratic mappings for which there exists a constant $\theta > 0$ such that f satisfies*

$$\| f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2(m^2-2)f(x) + 2f(y) \|_F$$
$$\leq \theta H(x,y), \quad x \perp y \qquad (2.15)$$

for all $(x,y) \in E^2, x \perp y$ and $H : E^2 \to \mathbb{R}^+ \cup \{0\}$ is a non negative real valued function, such that

$$R(x) = \sum_{j=0}^{\infty} \frac{H(m^j x, 0)}{m^{2j}} (< \infty)(m \neq 0) \qquad (2.16)$$

is a non negative function on x, with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ and the condition

$$\lim_{k\to\infty} \frac{H(m^k x, m^k y)}{m^{2k}} = 0 \qquad (2.17)$$

holds. Then there exists a unique orthogonally Euler - Lagrange quadratic mappings $Q : E \to F$ such that

$$\| f(x) - Q(x) \|_F \quad \leq \quad \frac{\theta}{2m^2} R(x) + \frac{||f(0)||_F}{|m^2-1|} \qquad (2.18)$$

for all $x \in E$. In addition $f : E \to F$ is a mapping such that the transformation $t \to f(tx)$ is continuous in real t for each fixed $x \in E$, then Q is $\mathbb{R}-$ linear mapping.

Proof. Letting $y = 0$ in (2.15), we get

$$\left\| \frac{f(mx)}{m^2} - f(x) + \frac{f(0)}{m^2} \right\|_F \leq \frac{\theta}{2\,m^2} H(x,0) \quad (m \neq 0)$$
$$\left\| f(x) - \frac{f(mx)}{m^2} \right\|_F \leq \frac{\theta}{2\,m^2} H(x,0) + \frac{||f(0)||_F}{m^2} \quad (m \neq 0) \qquad (2.19)$$

for all $x \in E$. Now replacing x by mx divide by m^2 in (2.19), we obtain

$$\left\|\frac{f(mx)}{m^2} - \frac{f(m^2x)}{m^4}\right\|_F \leq \frac{\theta}{2\,m^4}H(mx,0) + \frac{||f(0)||_F}{m^4}.$$

Using (2.19) and the above inequality, we arrive

$$\left\|f(x) - \frac{f(m^2x)}{m^4}\right\|_F \leq \frac{\theta}{2m^2}\left[H(x,0) + \frac{H(mx,0)}{m^2}\right] + \frac{||f(0)||_F}{m^2}\left[1 + \frac{1}{m^2}\right] \tag{2.20}$$

for all $x \in E$. Using the induction on n we obtain that

$$\left\|f(x) - \frac{f(m^nx)}{m^{2n}}\right\|_F \leq \frac{\theta}{2m^2}\sum_{j=0}^{n-1}\frac{H(m^jx,0)}{m^{2j}} + \frac{||f(0)||_F}{m^2}\sum_{j=0}^{n-1}\frac{1}{m^{2j}} \tag{2.21}$$

for all $x \in E$. In order to prove the convergence of the sequence $\{\frac{f(m^nx)}{m^{2n}}\}$ replace x by m^lx and divided by m^{2l} in (2.21), for any $n, l > 0$, we obtain

$$\begin{aligned}\left\|\frac{f(m^lx)}{m^{2l}} - \frac{f(m^{n+l}x)}{m^{2(n+l)}}\right\|_F &= \frac{1}{m^{2l}}\left\|f(m^lx) - \frac{f(m^{n+l}x)}{m^{2n}} - \right\|_F \\ &\leq \frac{\theta}{2m^2}\sum_{j=0}^{n-1}\frac{H(m^{j+l}x,0)}{m^{2(j+l)}} + \frac{||f(0)||_F}{m^2}\sum_{j=0}^{n-1}\frac{1}{m^{2(j+l)}} \\ &\to 0 \quad \text{as} \quad l \to \infty\end{aligned}$$

for all $x \in E$. Thus $\{\frac{f(m^nx)}{m^{2n}}\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ such that

$$Q(x) = \lim_{n\to\infty}\frac{f(m^nx)}{m^{2n}}, \quad \forall x \in E.$$

Letting $n \to \infty$ in (2.21) and using the definition of $Q(x)$ and (2.16), we arrive at the formula (2.18). Indeed

$$\begin{aligned}\|f(x) - Q(x)\|_F &\leq \frac{\theta}{2m^2}\sum_{j=0}^{\infty}\frac{H(m^jx,0)}{m^{2j}} + \frac{||f(0)||_F}{m^2}\sum_{j=0}^{\infty}\frac{1}{m^{2j}} \\ &\leq \frac{\theta}{2m^2}R(x) + \frac{||f(0)||_F}{m^2}\left[\frac{m^2}{m^2-1}\right] \\ &\leq \frac{\theta}{2m^2}R(x) + \frac{||f(0)||_F}{|m^2-1|}\end{aligned}$$

for all $x \in E$. To prove Q satisfies (1.4), replace $(x,\ y)$ by $(m^nx,\ m^ny)$ in (2.15) and divide by m^{2n} then it follows that

$$\begin{aligned}&\frac{1}{m^{2n}}\parallel f(m^n(mx+y)) + f(m^n(mx-y)) - 2f(m^n(x+y)) - 2f(m^n(x-y)) \\ &- 2(m^2-2)f(m^nx) + 2f(m^ny)\parallel_F \leq \frac{\theta}{m^{2n}}H(m^nx, m^ny)\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\parallel Q(mx+y) + Q(mx-y) - 2Q(x+y) - 2Q(x-y) - 2(m^2-2)Q(x) + 2Q(y)\parallel_F \leq 0.$$

which gives

$$Q(mx+y)+Q(mx-y)=2Q(x+y)+2Q(x-y)+2(m^2-2)Q(x)-2Q(y)$$

for all $x, y \in E$ with $x \perp y$. Therefore $Q : E \to F$ is an orthogonally Euler - Lagrange quadratic mapping which satisfies (1.4). To prove the uniqueness of Q, let Q' be another orthogonally Euler - Lagrange quadratic mapping satisfying (1.4) and the inequality (2.18). We have

$$\begin{aligned} \left\|Q(x)-Q'(x)\right\|_F = & \frac{1}{m^{2n}}\left\{\|Q(m^n x)-f(m^n x)\|_F+\left\|f(m^n x)-Q'(m^n x)\right\|_F\right\} \\ \leq & \frac{1}{m^{2n}}\left\{\frac{\theta}{m^2}R(x)+\frac{2||f(0)||_F}{|m^2-1|}\right\} \\ & \to 0 \quad \text{as} \quad n\to\infty \end{aligned}$$

for all $x \in E$. Therefore Q is unique. This completes the proof of the theorem. □

Theorem 2.4. *Let E be a real orthogonality normed linear space and F be a real complete normed linear space. Assume in addition that $f : E \to F$ is an approximately quadratic mappings for which there exists a constant $\theta > 0$ such that f satisfies*

$$\begin{aligned} &\| f(mx+y)+f(mx-y)-2f(x+y)-2f(x-y)-2(m^2-2)f(x)+2f(y) \|_F \\ &\leq \theta H(x,y), \quad x\perp y \end{aligned} \tag{2.22}$$

for all $(x, y) \in E^2, x \perp y$ and $H : E^2 \to \mathbb{R}^+ \cup \{0\}$ is a non negative real valued function, such that

$$R(x)=\sum_{j=0}^{\infty} m^{2j} H\left(\frac{x}{m^{j+1}},0\right)(<\infty)(m\neq 0) \tag{2.23}$$

is a non negative function on x, with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ and the condition

$$\lim_{k\to\infty} m^{2k} H\left(\frac{x}{m^k},\frac{y}{m^k}\right)=0 \tag{2.24}$$

holds. Then there exists a unique orthogonally Euler - Lagrange quadratic mappings $Q : E \to F$ such that

$$\| f(x)-Q(x) \|_F \leq \frac{\theta}{2}R(x)+\frac{||f(0)||_F}{|1-m^2|} \tag{2.25}$$

for all $x \in E$. In addition $f : E \to F$ is a mapping such that the transformation $t \to f(tx)$ is continuous in real t for each fixed $x \in E$, then Q is $\mathbb{R}-$ linear mapping.

Proof. Replacing x by $\frac{x}{m}$ in (2.19) and using the proof of Theorem 2.3, we arrive at the desired result. □

The following two analogous Theorems 2.5 and 2.6 can be obtained as two special cases: either $m = 1$ or $m = -1$. In these two cases the pertinent functional equations are obviously equivalent to the classical quadratic equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.26}$$

for all $x, y \in E$ with $x \perp y$.

Theorem 2.5. *Let $f : E \to F$ be a mapping satisfying the inequality*

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \|_F \quad \leq \epsilon \left[\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| x \|_E^{2p} \right) \right] \tag{2.27}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \tag{2.28}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique Euler - Lagrange quadratic mapping such that

$$\| f(x) - Q(x) \|_F \leq \frac{3\,\epsilon}{4 - 2^{2p}} \|x\|_E^{2p} \tag{2.29}$$

for all $x \in E$.

Proof. Letting $y = x$ in (2.27), we get

$$\left\| \frac{f(2x)}{4} - f(x) \right\|_F \leq \frac{3\,\epsilon}{4} \|x\|_E^{2p} \tag{2.30}$$

for all $x \in E$. Now Replacing x by $2x$ and dividing by 4 in (2.30) and summing the resultant inequality with (2.30), we arrive

$$\left\| \frac{f(2^2 x)}{4^2} - f(x) \right\|_F \leq \frac{3\,\epsilon}{4} \left(1 + \frac{2^{2p}}{4} \right) \|x\|_E^{2p} \tag{2.31}$$

for all $x \in E$. Using induction on n, we obtain that

$$\begin{aligned} \left\| \frac{f(2^n x)}{4^n} - f(x) \right\|_F &\leq \frac{3\,\epsilon}{4} \sum_{k=0}^{n-1} \frac{2^{2pk}}{4^k} \|x\|_E^{2p} \\ &\leq \frac{3\,\epsilon}{4} \sum_{k=0}^{\infty} \frac{2^{2pk}}{4^k} \|x\|_E^{2p} \end{aligned} \tag{2.32}$$

for all $x \in E$. In order to prove the convergence of the sequence $\{f(2^n x)/4^n\}$, replace x by $2^l x$ and divide by 4^l in (2.32), for $n, l > 0$,we obtain

$$\begin{aligned} \left\| \frac{f(2^{n+l} x)}{4^{l+n}} - \frac{f(2^l x)}{4^l} \right\|_F &= \frac{1}{4^l} \left\| \frac{f(2^{n+l} x)}{4^n} - f(2^l x) \right\|_F \\ &\leq \frac{1}{4^l} \frac{3\,\epsilon}{4} \sum_{k=0}^{n-1} \frac{2^{2pk}}{4^k} \left\| 2^l x \right\|_E^{2p} \\ &\leq \frac{3\,\epsilon}{4} \sum_{k=0}^{\infty} \frac{2^{2p(k+l)}}{4^{(k+l)}} \|x\|_E^{2p} \\ &\leq \frac{3\,\epsilon}{4} \sum_{k=0}^{\infty} \frac{1}{2^{2(1-p)(k+l)}} \|x\|_E^{2p}\,. \end{aligned} \tag{2.33}$$

As $p < 1$, the R.H.S of (2.33) tends to 0 as $l \to \infty$. Thus $\{f(2^n x)/4^n\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ and define

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \quad \forall x \in E.$$

Letting $n \to \infty$ in (2.32), we arrive the formula (2.29) for all $x \in E$. To prove Q satisfies (1.4) and it is unique the proof is similar to that of Theorem 2.1. Hence the proof is complete. □

Theorem 2.6. *Let $f : E \to F$ be a mapping satisfying the inequality*

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \|_F \quad \leq \epsilon \left[\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| x \|_E^{2p} \right) \right] \quad (2.34)$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon > 0$ and $p > 1$. Then the limit

$$Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \quad (2.35)$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler - Lagrange quadratic mapping such that

$$\| f(x) - Q(x) \|_F \leq \frac{3\,\epsilon}{2^{2p} - 4} \|x\|_E^{2p} \quad (2.36)$$

for all $x \in E$.

Proof. Replacing x by $\frac{x}{m}$ in (2.30) and using the proof of Theorem 2.5, we arrive at the desired result. □

References

[1] **J. Alonso and C. Benitez**, 1988, *Orthogonality in normed linear spaces: survey I. Main properties*, Extracta Math. 3, 1-15.

[2] **J. Alonso and C. Benitez**, 1989, *Orthogonality in normed linear spaces: a survey II. Relations between main orthogonalities*, Extracta Math. 4, 121-131.

[3] **J. Aczel and J. Dhombres**, 1989, *Functional Equations in Several Variables,* (Cambridge Univ,Press,Cambridge).

[4] **R. Ger and J. Sikorska**, 1995, *Stability of the orthogonal additivity*, Bull. Polish Acad. Sci. Math. 43, 143-151.

[5] **S. Gudder and D. Strawther**, 1975, *Orthogonally additive and orthogonally increasing functions on vector spaces*, Pacific J. Math. 58, 427-436.

[6] **D.H.Hyers**, 1941, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci., U.S.A., 27, 222-224.

[7] **J.C.James**, 1945, *Orthogonality in normed linear spaces,* Duck Math. J. 12, 291-302.

[8] **P. Nakmahachalasint**, 2007, *On the generalized Ulam - Gavruta - Rassias stability of a mixed type linear and Euler - Lagrange - Rassias functional equation*, International Journal of Mathematics and Mathematical Sciences: IJMMS, Volume 2007, Article ID 63239, 1-10, doi:10.1155/2007/63239, Hindawi Publ. Corporation., 2007.

[9] **P. Nakmahachalasint**, 2007, *Hyers-Ulam-Rassias and Ulam - Gavruta - Rassias stabilities of an additive functional equation in several variables*, International Journal of Mathematics and Mathematical Sciences: IJMMS, Hindawi Publ. Corporation., 2007.

[10] **A. Pietrzyk**, 2006, *Stability of the Euler - Lagrange - Rassias functional equation*, Demonstratio Mathematica, 39 (3), 523 - 530.

[11] **J.M.Rassias**, 1982, *On approximately of approximately linear mappings by linear mappings,* J. Funct. Anal. USA, 46, 126-130.

[12] **J.M.Rassias**, 1984, *On approximately of approximately linear mappings by linear mappings,* Bull. Sc. Math, 108, 445-446.

[13] **J. M. Rassias**, 1989, *Solution of problem of Ulam*, J. Approx. Th. USA, 57, 268-273.

[14] **J. M. Rassias**, 1992, *Solution of a stability problem of Ulam*, Discuss. Math. 12, 95-103.

[15] **J. M. Rassias**, 1992, *On the stability of the Euler- Lagrange functional equation*, Chinese J. Math. 20, 185-190.

[16] **J. M. Rassias**, 1994, *On the stability of the non-linear Euler- Lagrange functional equation in real normed linear spaces*, J. Math. Phys. Sci. 28, 231-235.

[17] **J. M. Rassias**, 1994, *Complete solution of the multi dimensional problem of Ulam*, Discuss Math. 14, 101-107.

[18] **J. M. Rassias**, 1996, *On the stability of the general Euler- Lagrange functional equation*, Demonstratio Math. 29, 755-766.

[19] **J. M. Rassias**, 1998, *Solution of the Ulam stability problem for Euler- Lagrange quadratic mappings*, J. Math. Anal. Appl. 220, 613-639.

[20] **J. M. Rassias**, 1999, *Generalizations of the Eulers theorem to Heptagons leading to a quadratic vector identity*. In Advances in Equations and Inequalities, (Hadronic press, Inc.), (1999) 179-183.

[21] **J. M. Rassias**, 1999, *On the stability of the multi dimensional Euler- Lagrange functional equation*, J. Ind. Math. Soc. 66, 1-9.

[22] **J. M. Rassias**, 2002, *On some approximately quadratic mappings being exactly quadratic*, J. Ind. Math. Soc 69, 155-160.

[23] **J. M. Rassias and M. J. Rassias**, 2005, *On the Ulam stability for Euler- Lagrange type quadratic functional equations*, Austral. J. Math. Anal. Appl. 2, 1-10.

[24] **Th.M.Rassias**, 1978, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc., 72, 297-300.

[25] **J. Rätz**, 1985, *On orthogonally additive mappings*, Aequationes Math. 28, 35-49.

[26] **K. Ravi and M. Arunkumar**, 2007, *On the Ulam- Gavruta- Rassias stability of the orthogonally Euler- Lagrange type functional equation*, IJAMAS, Vol. 7, No. Fe 07, 143-156.

[27] **S.M.Ulam**, 1960, *Problems in Modern Mathematics,* (Wiley, New York).

[28] **F. Vajzovic**, 1967, *Uber das Funktional H mit der Eigenschaft:* $(x, y) = 0 \Rightarrow H(x+y)+H(x-y) = 2H(x) + 2H(y)$, Glasnik Mat. Ser. III 2 (22), 73-81.