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*Research Article*

# Stability of General Newton Functional Equations for Logarithmic Spirals

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We investigate the generalized Hyers-Ulam stability of Newton functional equations for logarithmic spirals.



## 1. Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [1] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

*Let $G_1$ be a group and let $G_2$ be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

The case of approximately additive mappings was solved by Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. Later, the result of Hyers was significantly generalized for additive mappings by Aoki [3] and for linear mappings by Rassias [4]. It should be remarked that we can find in the books [5–7] a lot of references concerning the stability of functional equations.

Recently, Jung and Sahoo [8] proved the generalized Hyers-Ulam stability of the functional equation $f(\sqrt{r^2+1}) = f(r) + \arctan(1/r)$ which is closely related to the square root spiral, for the case that $f(1) = 0$ and $f(r)$ is monotone increasing for $r > 0$ (see [9, 10]).

By $\mathcal{F}$ we denote the set of all functions $f : (0,\infty)\to\mathbb{R}$. Let $\Delta$ be the difference operator defined by

$$(\Delta f)(r) = f(r+1) - f(r) \quad (r > 0) \tag{1.1}$$

for any $f \in \mathcal{F}$. Throughout this paper, let $n$ be a fixed positive integer, and we define an operator $\Delta^n : \mathcal{F}\to\mathcal{F}$ by

$$(\Delta^n f)(r) = \Delta(\Delta^{n-1} f)(r) \quad (r > 0) \tag{1.2}$$

for all $f \in \mathcal{F}$, where we set $\Delta^0 f = f$. For instance, we see that

$$\begin{aligned}(\Delta^2 f)(r) &= f(r+2) - 2f(r+1) + f(r),\\ (\Delta^3 f)(r) &= f(r+3) - 3f(r+2) + 3f(r+1) - f(r).\end{aligned} \tag{1.3}$$

In this paper, we will investigate the generalized Hyers-Ulam stability of the Newton difference (operator) equations

$$\Delta^n f(r) = A \ln R_n(r) \tag{1.4}$$

for all $r > 0$ and some fixed integer $n > 0$, where $A > 0$ is a constant and

$$R_1(r) = \frac{r+1}{r}, \qquad R_k(r) = \frac{R_{k-1}(r+1)}{R_{k-1}(r)} \tag{1.5}$$

for $k \in \{2, 3, \ldots, n\}$.

We will say that (1.4) has the generalized Hyers-Ulam stability whenever a (given) function $f : (0,\infty)\to\mathbb{R}$ satisfies the inequality

$$\left|\Delta^n f(r) - A \ln R_n(r)\right| \le \varphi_n(r) \tag{1.6}$$

for all $r > 0$, where $\varphi_n : (0,\infty)\to[0,\infty)$ is a given nonnegative function, there exists a solution of (1.4) which is not far from $f$.

## 2. Newton $n$-ary difference equation

The difference equation in (1.4) is called the Newton $n$-ary difference (operator) equation. In the following theorem, we give a partial solution to the generalized Hyers-Ulam stability problem of (1.4).

**Theorem 2.1.** *If a function $f : (0,\infty)\to\mathbb{R}$ satisfies the inequality* (1.6) *for all $r > 0$ and some integer $n > 0$, where $\varphi_n : (0,\infty)\to[0,\infty)$ is a function which satisfies*

$$\Phi_n(r) = \sum_{k=0}^{\infty} \varphi_n(r+k) < \infty \tag{2.1}$$

*for any $r > 0$, then there exists a unique function $F_n : (0,\infty)\to\mathbb{R}$ such that $\Delta F_n(r) = A \ln R_n(r)$ and*

$$\left|F_n(r) - \Delta^{n-1} f(r)\right| \le \Phi_n(r) \tag{2.2}$$

*for each $r > 0$.*

*Proof.* It follows from (1.6) that

$$\begin{aligned}
\left|\Delta^n f(r) - A\ln R_n(r)\right| &\le \varphi_n(r),\\
\left|\Delta^n f(r+1) - A\ln R_n(r+1)\right| &\le \varphi_n(r+1),\\
\vdots \qquad\qquad & \qquad \vdots\\
\left|\Delta^n f(r+m-1) - A\ln R_n(r+m-1)\right| &\le \varphi_n(r+m-1)
\end{aligned} \tag{2.3}$$

for any $r > 0$ and $m \in \mathbb{N}$. In view of triangular inequality, the above inequalities yield

$$\left|\sum_{k=0}^{m-1} \Delta^n f(r+k) - \sum_{k=0}^{m-1} A\ln R_n(r+k)\right| \le \sum_{k=0}^{m-1} \varphi_n(r+k). \tag{2.4}$$

Substitute $r+\ell$ for $r$ in (2.4) and then substitute $k$ for $k+\ell$ in the resulting inequality to obtain

$$\left|\sum_{k=\ell}^{\ell+m-1} \Delta^n f(r+k) - \sum_{k=\ell}^{\ell+m-1} A\ln R_n(r+k)\right| \le \sum_{k=\ell}^{\ell+m-1} \varphi_n(r+k) \tag{2.5}$$

for all $r > 0$ and $\ell, m \in \mathbb{N}$.

By some manipulation, we further have

$$\begin{aligned}
&\left|\sum_{k=0}^{\ell+m-1} \Delta^n f(r+k) - \sum_{k=0}^{\ell+m-1} A\ln R_n(r+k) + \Delta^{n-1} f(r)\right.\\
&\qquad \left. - \sum_{k=0}^{\ell-1} \Delta^n f(r+k) + \sum_{k=0}^{\ell-1} A\ln R_n(r+k) - \Delta^{n-1} f(r)\right| \le \sum_{k=\ell}^{\ell+m-1} \varphi_n(r+k)
\end{aligned} \tag{2.6}$$

for every $r > 0$ and $\ell, m \in \mathbb{N}$. Thus, considering (2.1), we see that the sequence

$$\left\{\sum_{k=0}^{m-1} \left[\Delta^n f(r+k) - A\ln R_n(r+k)\right] + \Delta^{n-1} f(r)\right\}_{n=1,2,3,\dots} \tag{2.7}$$

is a Cauchy sequence for any $r > 0$. Hence, we can define a function $F_n : (0,\infty) \to \mathbb{R}$ by

$$F_n(r) = \sum_{k=0}^{\infty} \left[\Delta^n f(r+k) - A\ln R_n(r+k)\right] + \Delta^{n-1} f(r). \tag{2.8}$$

By (2.8), we get

$$\begin{aligned}
\Delta F_n(r) &= F_n(r+1) - F_n(r)\\
&= \sum_{k=1}^{\infty} \left[\Delta^n f(r+k) - A\ln R_n(r+k)\right] + \Delta^{n-1} f(r+1)\\
&\qquad - \sum_{k=0}^{\infty} \left[\Delta^n f(r+k) - A\ln R_n(r+k)\right] - \Delta^{n-1} f(r)\\
&= A\ln R_n(r)
\end{aligned} \tag{2.9}$$

for all $r > 0$. In view of (2.1) and (2.8), if we let $m$ go to infinity in (2.4), then we obtain (2.2).

It only remains to prove the uniqueness of the function $F_n$. If a function $H : (0,\infty)\to\mathbb{R}$ satisfies $\Delta H(r) = A\ln R_n(r)$ for each $r > 0$, then we can easily show that

$$H(r+m) - H(r) = \sum_{k=0}^{m-1} A\ln R_n(r+k) \tag{2.10}$$

for all $r > 0$ and $m \in \mathbb{N}$. Now, assume that $G_n : (0,\infty)\to\mathbb{R}$ satisfies $\Delta G_n(r) = A\ln R_n(r)$ and the inequality (2.2) in place of $F_n$. By (2.1), (2.2), and (2.10), we obtain

$$\left|F_n(r) - G_n(r)\right| = \left|F_n(r+m) - G_n(r+m)\right| \le 2\Phi_n(r+m) \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{2.11}$$

for any $r > 0$, which proves the uniqueness of $F_n$. □

## 3. Application to logarithmic spirals

For given $\alpha > 1$ and $c > 0$, the equation

$$r = ce^{\theta/\sqrt{\alpha^2-1}} \tag{3.1}$$

represents a logarithmic spiral in the polar coordinates $(r,\theta)$. We know that this formula is equivalent to

$$\theta = \sqrt{\alpha^2-1}(\ln r - \ln c). \tag{3.2}$$

Let us define $f(r) = \sqrt{\alpha^2-1}(\ln r - \ln c)$ so that we can write the above expression in a simpler form, $\theta = f(r)$. Then $f$ is a solution of (1.4) for $n = 1$ and $A = \sqrt{\alpha^2-1}$, that is, $f$ is a solution of the equation

$$\Delta f(r) = \sqrt{\alpha^2-1}\ln\frac{r+1}{r}, \tag{3.3}$$

which may be called the equation for logarithmic spirals.

We will now solve (3.3) by using [9, Theorem 1].

**Theorem 3.1.** *If a function $f : (0,\infty)\to\mathbb{R}$ satisfies* (3.3), *then there exists a periodic function $\sigma : \mathbb{R}\to\mathbb{R}$ of period 1 such that*

$$f(r) = \sigma(r) + \sqrt{\alpha^2-1}\ln r \tag{3.4}$$

*for all $r > 0$.*

*Proof.* If we set

$$\psi(r) = \sqrt{\alpha^2-1}\ln\frac{r+1}{r} \tag{3.5}$$

for all $r > 0$, then we have

$$\psi(r+s) - \psi(r) = \sqrt{\alpha^2-1}\ln\frac{r^2+(s+1)r}{r^2+(s+1)r+s} < 0 \tag{3.6}$$

for any $r, s > 0$, which implies that $\varphi$ is monotonically decreasing. Moreover, we also see that

$$\lim_{r\to\infty}\varphi(r) = \sqrt{\alpha^2-1}\lim_{r\to\infty}\ln\frac{r+1}{r} = 0. \tag{3.7}$$

According to [9, Theorem 1], the general solution of (3.3) is given by

$$f(r) = \sigma(r) + \sum_{k=0}^{\infty}[\varphi(k+1) - \varphi(r+k)] = \sigma(r) + \sqrt{\alpha^2-1}\ln r, \tag{3.8}$$

where $\sigma$ is an arbitrary periodic function of period 1. □

If we set $n = 1$ in Theorem 2.1 and apply Theorem 3.1, then we get the following corollary concerning the generalized Hyers-Ulam stability of (3.3).

**Corollary 3.2.** *If a given function* $f : (0,\infty)\to\mathbb{R}$ *satisfies the inequality*

$$\left|\Delta f(r) - \sqrt{\alpha^2-1}\ln\frac{r+1}{r}\right| \le \varphi(r) \tag{3.9}$$

*for all* $r > 0$ *and some* $\alpha > 1$, *where* $\varphi : (0,\infty)\to[0,\infty)$ *is a function which satisfies the condition*

$$\Phi(r) = \sum_{k=0}^{\infty}\varphi(r+k) < \infty \tag{3.10}$$

*for any* $r > 0$, *then there exists a unique periodic function* $\sigma : \mathbb{R}\to\mathbb{R}$ *of period 1 such that*

$$\left|f(r) - \sigma(r) - \sqrt{\alpha^2-1}\ln r\right| \le \Phi(r) \tag{3.11}$$

*for all* $r > 0$.

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