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Refined Hyers–Ulam approximation of approximately Jensen type mappings

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Abstract

In 1940 S.M. Ulam proposed the famous Ulam stability problem. In 1941 D.H. Hyers solved this problem for additive mappings subject to the Hyers condition on approximately additive mappings. In this paper we generalize the Hyers result for the Ulam stability problem for Jensen type mappings, by considering approximately Jensen type mappings satisfying conditions weaker than the Hyers condition, in terms of products of powers of norms. This process leads to a refinement of the well-known Hyers–Ulam approximation for the Ulam stability problem. Besides we introduce additive mappings of the first and second form and investigate pertinent stability results for these mappings. Also we introduce approximately Jensen type mappings and prove that these mappings can be exactly Jensen type, respectively. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology.

Résumé

En 1940 S.M. Ulam proposés le problème célèbre de stabilité d' Ulam. En 1941 D.H. Hyers a résolu ce problème pour les tracés additifs sujet à la condition de Hyers sur les tracés approximativement additifs. Dans cet article nous généralisons le résultat de Hyers pour le problème de stabilité d' Ulam pour le type tracés de Jensen, en considérant approximativement le type tracés de Jensen satisfaisant des conditions plus faibles que la condition de Hyers, en termes de produits des puissances des normes. Ce processus mène à une amélioration de l'approximation bien connue de Hyers–Ulam pour le problème de stabilité d' Ulam. Sans compter que nous présentons les tracés additifs de la première et deuxième forme et étudions des résultats convenables de stabilité pour ces tracés. En outre nous présentons approximativement le type tracés de

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Jensen et montrons que ces tracés peuvent être exactement type de Jensen, respectivement. Ces résultats de stabilité peuvent être appliqués dans l'analyse stochastique, mathématiques financières et actuarielles, aussi bien qu'en la psychologie et la sociologie.

MSC: primary 39B; secondary 26D

Keywords: Ulam stability; Hyers condition; Refined approximation; Jensen type

1. Introduction

In 1940 and in 1964 S.M. Ulam [26] proposed the famous *Ulam stability problem*:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In 1941 D.H. Hyers [13] solved this stability problem for additive mappings subject to *the Hyers condition*

$$\left\| f(x_1 + x_2) - f(x_1) - f(x_2) \right\| \leqslant \delta \tag{$*$}$$

on approximately additive mappings $f : X \to Y$, for a fixed $\delta \geqslant 0$, and all $x_1, x_2 \in X$, where X is a real normed space and Y a real Banach space.

In 1951 D.G. Bourgin [3] was the second author to treat the Ulam problem for additive mappings. In 1978, according to P.M. Gruber [12], this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1980 and in 1987, I. Fenyö [7,8] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Z. Gajda and R. Ger [10] showed that one can get analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors: J. Aczél [1], C. Borelli and G.L. Forti [2,9], P.W. Cholewa [4], St. Czerwik [5], and H. Drljevic [6]. In 1982–2005 J.M. Rassias [16–21,23–25] and in 2003 M.J. Rassias and the author [22,25] solved the above Ulam problem for Jensen and Euler–Lagrange type mappings. In 1999 P. Gavruta [11] answered a question of ours [18] concerning the stability of the Cauchy equation. In 1998 S.-M. Jung [14] and in 2002–2003 M.J. Rassias and the author [21,22] investigated the Hyers–Ulam stability for additive and quadratic mappings on restricted domains. In this paper we generalize the Hyers result for the Ulam stability problem, by considering approximately Jensen and Jensen type mappings satisfying conditions weaker than the Hyers condition on approximately Jensen and Jensen type mappings, in terms of products of powers of norms. Besides we introduce additive mappings of the first and second form and investigate pertinent stability results for these mappings. Also we introduce approximately Jensen and Jensen type mappings being exactly Jensen and Jensen type, respectively. These stability results can be applied in stochastic analysis [15], financial and actuarial mathematics, as well as in psychology and sociology.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations. Besides let us denote with $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of natural numbers and $\mathbb{R}$ the set of real numbers.

Definition 1.1. A mapping $A : X \to Y$ is called *additive of the first form* if A satisfies the functional equation

$$A(x_1 + x_2) + A(x_1 - x_2) = 2A(x_1) \tag{1.1}$$

for all $x_1, x_2 \in X$ and $A(0) = 0$. We note that (1.1) is equivalent to *the Jensen equation*

$$A\left(\frac{x+y}{2}\right) = \frac{1}{2}\big[A(x) + A(y)\big], \tag{1.1a}$$

or

$$2A\left(\frac{x+y}{2}\right) = A(x) + A(y) \tag{1.1b}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called *Jensen mapping* if A satisfies the functional equation (1.1a) (or (1.1b)) and $A(0) = 0$.

Definition 1.2. A mapping $A : X \to Y$ is called *additive of the second form* if A satisfies the functional equation

$$A(x_1 + x_2) - A(x_1 - x_2) = 2A(x_2) \tag{1.2}$$

for all $x_1, x_2 \in X$. We note that (1.2) is equivalent to *the Jensen type equation*

$$A\left(\frac{x-y}{2}\right) = \frac{1}{2}\big[A(x) - A(y)\big], \tag{1.2a}$$

or

$$2A\left(\frac{x-y}{2}\right) = A(x) - A(y) \tag{1.2b}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called *Jensen type mapping* if A satisfies the functional equations (1.2a) (or (1.2b)).

Definition 1.3. A mapping $A : X \to Y$ is called *additive* if A satisfies the functional equation

$$A(x_1 + x_2) = A(x_1) + A(x_2) \tag{1.3}$$

for all $x_1, x_2 \in X$. We note that Eq. (1.3) is called also *Cauchy additive*.

Definition 1.4. A mapping $f : X \to Y$ is called *approximately odd* if f satisfies the functional inequality

$$\big\| f(x) + f(-x) \big\| \leqslant \theta \tag{1.4}$$

for some fixed $\theta \geqslant 0$ and for all $x \in X$.

In 1982 [16], 1984 [17], 1989 [18] and 1994 [19], we introduced and proved the following Theorem 1.1 for the complete solution of the Ulam stability problem for additive mappings satisfying a condition weaker than *the Hyers condition* $(*)$ [13] on approximately additive mappings, in terms of *a product of powers of norms*.

Theorem 1.1. *If a mapping $f : X \to Y$ satisfies the approximately additive inequality*

$$\big\| f(x_1 + x_2) - f(x_1) - f(x_2) \big\| \leqslant \delta \|x_1\|^{\alpha} \|x_2\|^{\beta}, \tag{1.5}$$

for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho = \alpha + \beta \in \mathbb{R}, \rho \neq 1$, and $\delta \geqslant 0$, and all $x_1, x_2 \in X$, then there exists a unique additive mapping $A : X \to Y$, which satisfies the formula

$$A(x) = \lim_{n\to\infty} \begin{cases} 2^{-n} f(2^n x), & \text{if } -\infty < \rho < 1, \\ 2^n f(2^{-n} x), & \text{if } \rho > 1 \end{cases}$$

and the inequality

$$\| f(x) - A(x) \| \leqslant \frac{\delta}{|2 - 2^\rho|} \|x\|^\rho$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

We note that *the Hyers condition* $(*)$ on approximately additive mappings is the corresponding inequality (1.5), when $\alpha = \beta = 0$.

2. Stability of the additive equation (1.1) of the first form

We introduce and prove the following new stability theorem 2.1 for additive mappings of the first form.

Theorem 2.1. *If a mapping $f : X \to Y$ satisfies the approximately additive inequality*

$$\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) \| \leqslant \delta \|x_1\|^\alpha \|x_2\|^\beta, \tag{2.1}$$

of the first form for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho = \alpha + \beta \in \mathbb{R}, \rho \neq 1$, and $\delta \geqslant 0$, and all $x_1, x_2 \in X$, then there exists a unique additive mapping $A : X \to Y$ of the first form, which satisfies the formula

$$A(x) = \lim_{n\to\infty} \begin{cases} 2^{-n} f(2^n x), & \text{if } -\infty < \rho < 1, \\ 2^n f(2^{-n} x), & \text{if } \rho > 1 \text{ and } f(0) = 0 \end{cases}$$

and the inequality

$$\| f(x) - A(x) \| \leqslant \begin{cases} \| f(0) \| + \frac{\delta}{2-2^\rho} \|x\|^\rho, & \text{if } -\infty < \rho < 1, \\ \frac{\delta}{2^\rho - 2} \|x\|^\rho, & \text{if } \rho > 1 \text{ and } f(0) = 0 \end{cases} \tag{2.2}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

We note that *the Hyers condition* $(*)$ on approximately additive mappings of the first form is the corresponding inequality (2.1), when $\alpha = \beta = 0$.

Proof. Substituting $x_1 = 0, x_2 = x$ in the inequality (2.1), one gets

$$f(-x) = -f(x) + 2f(0), \tag{2.3}$$

for all $x \in X$. Besides replacing $x_1 = x_2 = x$ in the inequality (2.1) and then employing the triangle inequality, one obtains the basic inequality

$$\| f(x) - 2^{-1} f(2x) \| \leqslant \frac{1}{2} \| f(0) \| + \frac{1}{2} \delta \|x\|^\rho, \tag{2.4}$$

for some $\delta \geqslant 0$, $-\infty < \rho < 1$, and all $x \in X$. Then by the triangle inequality, and *without induction*, we establish the general inequality

$$\begin{aligned}\left\| f(x) - 2^{-n} f\left(2^n x\right)\right\| &\leqslant \left\| f(x) - 2^{-1} f(2x)\right\| + 2^{-1}\left\| f(2x) - 2^{-1} f\left(2^2 x\right)\right\| + \cdots \\ &\quad + 2^{-(n-1)}\left\| f\left(2^{n-1}x\right) - 2^{-1} f\left(2^n x\right)\right\| \\ &= \frac{1}{2}\left\{\left[1 + 2^{-1} + \cdots + 2^{-(n-1)}\right]\left\| f(0)\right\| \right. \\ &\quad \left. + \left[1 + 2^{\rho-1} + \cdots + 2^{(n-1)(\rho-1)}\right]\delta\|x\|^{\rho}\right\} \\ &= \left(1 - 2^{-n}\right)\left\| f(0)\right\| + \frac{1 - 2^{n(\rho-1)}}{2 - 2^{\rho}}\delta\|x\|^{\rho} \end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}$. Thus from this inequality (2.5) and letting $n \to \infty$, we get the inequality

$$\left\| f(x) - A(x)\right\| \leqslant \left\| f(0)\right\| + \frac{\delta}{2 - 2^{\rho}}\|x\|^{\rho}, \quad \text{if } -\infty < \rho < 1. \tag{2.6}$$

We easily prove as in [16–19] that the formula

$$A(x) = 2^{-n} A\left(2^n x\right)$$

holds for any $n \in \mathbb{N}$, and all $x \in X$.

It is clear that for $n \geqslant m > 0$, we have

$$\left\|2^{-n} f\left(2^n x\right) - 2^{-m} f\left(2^m x\right)\right\| < \left(\delta + \left\| f(0)\right\|\right) \cdot 2^{-m} \to 0, \quad \text{as } m \to \infty.$$

Therefore we may apply a direct method to the definition of A, such that the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f\left(2^n x\right)$$

holds for all $x \in X$ [16–19]. From this formula and the inequality (2.1), it follows that $A : X \to Y$ is an additive mapping of the first form.

The proof of the uniqueness of $A : X \to Y$ and the last assertion in our Theorem 2.1 is obvious according to our works [16–19].

Similarly we prove the other part for $\rho > 1$ and $f(0) = 0$. In fact, setting $x_1 = x_2 = x/2$ in the inequality (2.1) and assuming $f(0) = 0$, we establish the other basic inequality

$$\left\| f(x) - 2f\left(2^{-1}x\right)\right\| \leqslant \frac{1}{2}\delta 2^{1-\rho}\|x\|^{\rho}. \tag{2.7}$$

Then we find the other general inequality

$$\begin{aligned}\left\| f(x) - 2^n f\left(2^{-n}x\right)\right\| &\leqslant \left\| f(x) - 2^1 f\left(2^{-1}x\right)\right\| + 2^1\left\| f\left(2^{-1}x\right) - 2^2 f\left(2^{-2}x\right)\right\| + \cdots \\ &\quad + 2^{n-1}\left\| f\left(2^{-(n-1)}x\right) - 2^1 f\left(2^{-n}x\right)\right\| \\ &= \frac{1}{2}2^{1-\rho}\left[1 + 2^{1-\rho} + \cdots + 2^{(n-1)(1-\rho)}\right]\delta\|x\|^{\rho} \\ &= \frac{1 - 2^{n(1-\rho)}}{2^{\rho} - 2}\delta\|x\|^{\rho} \end{aligned} \tag{2.8}$$

for all $n \in \mathbb{N}$. Thus from this inequality (2.8) and the formula

$$A(x) = \lim_{n\to\infty} 2^n f\left(2^{-n}x\right),$$

and letting $n \to \infty$,we get the inequality

$$\|f(x) - A(x)\| \leqslant \frac{\delta}{2^{\rho} - 2}\|x\|^{\rho}, \quad \text{if } \rho > 1, \tag{2.9}$$

and if $f(0) = 0$. Therefore the proof of the inequality (2.2) is complete.

The rest of the proof is omitted as similar to the corresponding proof in our theorems [16–19]. $\square$

Corollary 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and the approximately additive inequality*

$$\|f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1)\| \leqslant \delta\|x_1\|^{\alpha}\|x_2\|^{\beta},$$

of the first form for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho = \alpha + \beta \in \mathbb{R}$, $\rho \neq 1$, and $\delta \geqslant 0$, and all $x_1, x_2 \in X$, then there exists a unique additive mapping $A : X \to Y$ of the first form, which satisfies the formula

$$A(x) = \lim_{n\to\infty} \begin{cases} 2^{-n} f(2^n x), & \text{if } -\infty < \rho < 1, \\ 2^n f(2^{-n} x), & \text{if } \rho > 1 \end{cases}$$

and the inequality

$$\|f(x) - A(x)\| \leqslant \frac{\delta}{|2 - 2^{\rho}|}\|x\|^{\rho}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

3. Stability of the additive equation (1.2) of the second form

We introduce and prove the following new stability Theorem 3.1 for additive mappings of the second form.

Theorem 3.1. *If a mapping $f : X \to Y$ satisfies the approximately additive inequality*

$$\|f(x_1 + x_2) - f(x_1 - x_2) - 2f(x_2)\| \leqslant \delta\|x_1\|^{\alpha}\|x_2\|^{\beta} \tag{3.1}$$

of the second form for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho = \alpha + \beta \in \mathbb{R}$, $\rho \neq 1$, and $\delta \geqslant 0$ and for all $x_1, x_2 \in X$, then there exists a unique additive mapping $A : X \to Y$ of the second form, which satisfies the formula

$$A(x) = \lim_{n\to\infty} \begin{cases} 2^{-n} f(2^n x), & \text{if } -\infty < \rho < 1, \\ 2^n f(2^{-n} x), & \text{if } \rho > 1 \end{cases}$$

and the inequality

$$\|f(x) - A(x)\| \leqslant \frac{\delta}{|2 - 2^{\rho}|}\|x\|^{\rho} \tag{3.2}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

We note that *the Hyers condition* $(*)$ on approximately additive mappings of the second form is the corresponding inequality (3.1), when $\alpha = \beta = 0$.

Proof. Replacing $x_1 = x_2 = 0$ in (3.1), we find

$$f(0) = 0. \tag{3.3}$$

Substituting $x_1 = 0, x_2 = x$ in the inequality (3.1), one gets $f(-x) = -f(x)$.

Besides, substituting $x_1 = x_2 = x$ in (3.1), one gets

$$\left\| f(x) - 2^{-1} f(2x) \right\| \leqslant \frac{1}{2} \delta \|x\|^\rho, \tag{3.4}$$

for some $\delta \geqslant 0$, $-\infty < \rho < 1$, and all $x \in X$. Therefore from (3.4) and the triangle inequality, we obtain

$$\begin{aligned} \left\| f(x) - 2^{-n} f\left(2^n x\right) \right\| &\leqslant \left\| f(x) - 2^{-1} f(2x) \right\| + 2^{-1} \left\| f(2x) - 2^{-1} f\left(2^2 x\right) \right\| + \cdots \\ &\quad + 2^{-(n-1)} \left\| f\left(2^{n-1} x\right) - 2^{-1} f\left(2^n x\right) \right\| \\ &= \frac{1}{2} \left[1 + 2^{\rho-1} + \cdots + 2^{(n-1)(\rho-1)}\right] \delta \|x\|^\rho \\ &= \frac{1 - 2^{n(\rho-1)}}{2 - 2^\rho} \delta \|x\|^\rho \end{aligned} \tag{3.5}$$

for some $\delta \geqslant 0$, $-\infty < \rho < 1$, any $n \in \mathbb{N}$, and all $x \in X$.

We easily prove as in [16–19] that

$$A(x) = 2^{-n} A\left(2^n x\right)$$

holds for any $n \in \mathbb{N}$, and all $x \in X$.

It is clear that for $n \geqslant m > 0$, we have

$$\left\| 2^{-n} f\left(2^n x\right) - 2^{-m} f\left(2^m x\right) \right\| < \left(\delta + \left\| f(0) \right\|\right) \cdot 2^{-m} \to 0, \quad \text{as } m \to \infty.$$

Therefore we may apply a direct method to the definition of A, such that the formula

$$A(x) = \lim_{n \to \infty} 2^{-n} f\left(2^n x\right)$$

holds for all $x \in X$ [16–19]. From this formula and the inequality (3.1), it follows that $A : X \to Y$ is an additive mapping of the second form.

The proof of the uniqueness of $A : X \to Y$ and the last assertion in our Theorem 3.1 is obvious according to our works [16–19].

Similarly we prove the other part for $\rho > 1$ and $f(0) = 0$. In fact, setting $x_1 = x_2 = x/2$ in the inequality (3.1), we establish the other basic inequality

$$\left\| f(x) - 2 f\left(2^{-1} x\right) \right\| \leqslant \frac{1}{2} \delta 2^{1-\rho} \|x\|^\rho. \tag{3.6}$$

Then we find the other general inequality

$$\begin{aligned} \left\| f(x) - 2^n f\left(2^{-n} x\right) \right\| &\leqslant \left\| f(x) - 2^1 f\left(2^{-1} x\right) \right\| + 2^1 \left\| f\left(2^{-1} x\right) - 2^2 f\left(2^{-2} x\right) \right\| + \cdots \\ &\quad + 2^{n-1} \left\| f\left(2^{-(n-1)} x\right) - 2^1 f\left(2^{-n} x\right) \right\| \\ &= \frac{1}{2} 2^{1-\rho} \left[1 + 2^{1-\rho} + \cdots + 2^{(n-1)(1-\rho)}\right] \delta \|x\|^\rho \\ &= \frac{1 - 2^{n(1-\rho)}}{2^\rho - 2} \delta \|x\|^\rho \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$. Thus from this inequality and the formula

$$A(x) = \lim_{n\to\infty} 2^n f\left(2^{-n}x\right),$$

and letting $n \to \infty$, we get the inequality

$$\left\| f(x) - A(x)\right\| \leqslant \frac{\delta}{2^\rho - 2}\|x\|^\rho, \quad \text{if } \rho > 1. \tag{3.8}$$

Therefore the proof of the inequality (3.2) is complete.

The rest of the proof is omitted as similar to the corresponding proof in our theorems [16–19]. □

4. Superstability of the Jensen equation (1.1b)

A functional equation $E(f) = 0$ is called superstable if every approximate solution of this equation is an exact (or genuine) solution.

We establish the following new superstability theorem 4.1 and superstability corollary 4.1 for Jensen mappings.

Theorem 4.1. *If a mapping $f : X \to Y$ satisfies the approximately Jensen inequality*

$$\left\|2f\left(\frac{x_1+x_2}{2}\right) - f(x_1) - f(x_2)\right\| \leqslant \delta\|x_1\|^\alpha\|x_2\|^\beta, \tag{4.1}$$

for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho = \alpha + \beta \in \mathbb{R}$, $\rho \neq 1$, and $\delta \geqslant 0$ and for all $x_1, x_2 \in X$, then there exists a unique Jensen mapping $A : X \to Y$, satisfying the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f\left(2^n x\right)$$

and the equation

$$f(x) - A(x) = f(0) \tag{4.2}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

We note that *the Hyers condition* $(*)$ on approximately Jensen mappings is the corresponding inequality (4.1), when $\alpha = \beta = 0$.

Proof. Setting $x_1 = x$, $x_2 = -x$ in the inequality (4.1), we find

$$\left\| f(-x) + f(x)\right\| \leqslant 2\left\| f(0)\right\| + \delta\|x\|^\rho.$$

Substituting $x_1 = 2x$ and $x_2 = 0$ in (4.1), one gets

$$f(x) - 2^{-1}f(2x) = \frac{1}{2}f(0), \tag{4.3}$$

for all $x \in X$. Therefore, we obtain

$$f(x) - 2^{-n}f\left(2^n x\right) = \left(1 - 2^{-n}\right)f(0), \tag{4.4}$$

for any $n \in \mathbb{N}$, and all $x \in X$. The rest of the proof is omitted as similar to the proof of Theorem 3.1. □

Corollary 4.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and the approximately Jensen inequality*

$$\left\| 2f\left(\frac{x_1 + x_2}{2}\right) - f(x_1) - f(x_2) \right\| \leqslant \delta \|x_1\|^{\alpha} \|x_2\|^{\beta}, \tag{4.1}$$

for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho = \alpha + \beta \in \mathbb{R}$, $\rho \neq 1$, and $\delta \geqslant 0$ and for all $x_1, x_2 \in X$, then there exists a unique Jensen mapping $A : X \to Y$, satisfying the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f\left(2^n x\right)$$

and the equation

$$f(x) = A(x) \tag{4.2}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

From (4.1)–(4.2) with $f(0) = 0$ we note that there exist approximately Jensen mappings $f : X \to Y$ which can be exactly Jensen mappings $A : X \to Y$. We define an equation $E(A) = 0$ $(**)$ as *superstable* if every approximate solution of $(**)$ is an exact solution.

5. Superstability of the Jensen type equation (1.2b)

We establish the following new superstability theorem 5.1 for Jensen type mappings.

Theorem 5.1. *If a mapping $f : X \to Y$ satisfies the approximately Jensen type inequality*

$$\left\| 2f\left(\frac{x_1 - x_2}{2}\right) - f(x_1) + f(x_2) \right\| \leqslant \delta \|x_1\|^{\alpha} \|x_2\|^{\beta}, \tag{5.1}$$

for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho = \alpha + \beta \in \mathbb{R}$, $\rho \neq 1$, and $\delta \geqslant 0$ and for all $x_1, x_2 \in X$, then there exists a unique Jensen type mapping $A : X \to Y$, satisfying the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f\left(2^n x\right)$$

and the equation

$$f(x) = A(x) \tag{5.2}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

We note that *the Hyers condition* $(*)$ on approximately Jensen type mappings is the corresponding inequality (5.1), when $\alpha = \beta = 0$.

Besides from (5.1)–(5.2) we note that there exist approximately Jensen type mappings $f : X \to Y$ which can be exactly Jensen type mappings $A : X \to Y$.

Proof. Replacing $x_1 = x_2 = 0$ in the inequality (5.1), we find $f(0) = 0$. Thus setting $x_1 = x, x_2 = -x$ in (5.1), one finds

$$\left\| f(-x) + f(x) \right\| \leqslant \delta \|x\|^{\rho}.$$

Substituting $x_1 = 2x$ and $x_2 = 0$ in (5.1), one gets $f(x) = 2^{-1} f(2x)$, and thus by (or without) induction on $n \in \mathbb{N}$, we obtain

$$f(x) = 2^{-n} f\left(2^n x\right), \tag{5.3}$$

for any $n \in \mathbb{N}$, and all $x \in X$. The rest of the proof is omitted as similar to the proof of Theorems 3.1 and 4.1. $\square$

References

[1] J. Aczél, Lectures on Functional Equations and their Applications, Academic Press, New York and London, 1966.
[2] C. Borelli, G.L. Forti, On a general Hyers–Ulam stability result, Internat. J. Math. Sci. 18 (1995) 229–236.
[3] D.G. Bourgin, Classes of transformations and bordering transformations, Bull. Amer. Math. Soc. 57 (1951) 223–237.
[4] P.W. Cholewa, Remarks on the stability of functional equations, Aequationes Math. 27 (1984) 76–86.
[5] St. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg 62 (1992) 59–64.
[6] H. Drljevic, On the stability of the functional quadratic on A-orthogonal vectors, Publ. Inst. Math. (Beograd) (N.S.) 36 (50) (1984) 111–118.
[7] I. Fenyö, Osservazioni su alcuni teoremi di D.H. Hyers, Istit. Lombardo Accad. Sci. Lett. Rend. A 114 (1980) (1982) 235–242.
[8] I. Fenyö, On an inequality of P.W. Cholewa, in: General Inequalities, vol. 5, in: Internat. Schriftenreiche Numer. Math., vol. 80, Birkhäuser, Basel–Boston, MA, 1987, pp. 277–280.
[9] G.L. Forti, Hyers–Ulam stability of functional equations in several variables, Aequationes Math. 50 (1995) 143–190.
[10] Z. Gajda, R. Ger, Subadditive multifunctions and Hyers–Ulam stability, in: General Inequalities, vol. 5, in: Internat. Schriftenreiche Numer. Math., vol. 80, Birkhäuser, Basel–Boston, MA, 1987.
[11] P. Gavruta, An answer to a question of John M. Rassias concerning the stability of Cauchy equation, in: Advances in Equations and Inequalities, Hadronic Math. Series, USA, 1999, pp. 67–71.
[12] P.M. Gruber, Stability of isometries, Trans. Amer. Math. Soc. USA 245 (1978) 263–277.
[13] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. 27 (1941) 222–224;
D.H. Hyers, The stability of homomorphisms and related topics, in: Global Analysis – Analysis on Manifolds, in: Texte zur Mathematik, vol. 57, Teubner, 1983, pp. 140–153.
[14] S.-M. Jung, On the Hyers–Ulam stability of the functional equations that have the quadratic property, J. Math. Anal. Appl. 222 (1998) 126–137.
[15] P. Malliavin, Stochastic Analysis, Springer, Berlin, 1997.
[16] J.M. Rassias, On approximation of approximately linear mappings by linear mappings, J. Funct. Anal. 46 (1982) 126–130.
[17] J.M. Rassias, On approximation of approximately linear mappings by linear mappings, Bull. Sci. Math. 108 (1984) 445–446.
[18] J.M. Rassias, Solution of a problem of Ulam, J. Approx. Theory 57 (1989) 268–273.
[19] J.M. Rassias, Complete solution of the multi-dimensional problem of Ulam, Discuss. Mathem. 14 (1994) 101–107.
[20] J.M. Rassias, Solution of the Ulam stability problem for Euler–Lagrange quadratic mappings, J. Math. Anal. Appl. 220 (1998) 613–639.
[21] J.M. Rassias, On the Ulam stability of mixed type mappings on restricted domains, J. Math. Anal. Appl. 276 (2002) 747–762.
[22] J.M. Rassias, M.J. Rassias, On the Ulam stability of Jensen and Jensen type mappings on restricted domains, J. Math. Anal. Appl. 281 (2003) 516–524.
[23] J.M. Rassias, Asymptotic behavior of mixed type functional equations, Austral. J. Math. Anal. Appl. 1 (1) (2004) 1–21.
[24] J.M. Rassias, The Ulam stability problem in approximation of approximately quadratic mappings by quadratic mappings, J. Ineq. Pure Appl. Math. 5 (3) (2004) 1–9.
[25] J.M. Rassias, M.J. Rassias, Asymptotic behavior of alternative Jensen and Jensen type functional equations, Bull. Sci. Math. 129 (7) (2005) 545–558.
[26] S.M. Ulam, Problems in Modern Mathematics, Wiley-Interscience, New York, 1964 (Chapter VI).