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Alternative Contraction Principle and Alternative Jensen and Jensen Type Mappings

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ABSTRACT

In 1940 and 1960 (and 1964) S. M. Ulam ([11],[12]) proposed the general Ulam stability problem: " When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true ?" Almost all proofs on the Ulam stability of additive functional equations have employed the well-known Hyers (direct) method established by D. H. Hyers [2] in 1941. On the other hand, V. Radu [6] in 2003 noticed that a fixed point alternative method is very important for the solution of the Ulam problem. Then, L. Cadariu and V. Radu [1] investigated the stability of the Jensen functional equation. In 2003-2004 M. J. Rassias and the author ([7]-[9]) of this paper investigated the Ulam stability of Jensen and Jensen type mappings by applying the Hyers method. An example of a stable functional equation with the Hyers method not to work, has recently been published by Z. Kaizer and Zs. Pales ([3],[5]). In this paper we establish the Ulam stability of alternative Jensen and Jensen type mappings as well as alternative additive mappings of two forms, via an alternative contraction principle. This kind of stability problems can be applied in stochastic analysis, actuarial and financial mathematics.

Key words and phrases

Alternative contraction principle, Alternative Jensen and Jensen type, Alternative additive mapping of two forms, Hyers (direct) method, Ulam stability.

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1. INTRODUCTION

Let X and Y be real linear spaces and $\mathbb{N}$ the set of natural numbers.
Definition 1.1. A mapping $A : X \rightarrow Y$ is called *alternative additive of the first form* if A satisfies the functional equation

(1.1) $$A(x_1 + x_2) + A(x_1 - x_2) = -2A(-x_1)$$

for all $x_1, x_2 \in X$. We note that (1.1) is equivalent to *the alternative Jensen equation*

$A\left(-\frac{x+y}{2}\right) = -\frac{1}{2}\left[A(x)+A(y)\right]$, or

(1.2) $$2A(-\frac{x+y}{2}) = -[A(x)+A(y)]$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called *alternative Jensen mapping* if A satisfies the functional equation (1.2).

Definition 1.2. A mapping $A : X \to Y$ is called *alternative additive of the second form* if A satisfies the functional equation

(1.3) $$A(x_1 + x_2) - A(x_1 - x_2) = -2A(-x_2)$$

for all $x_1, x_2 \in X$. We note that (1.3) is equivalent to *the alternative Jensen type equation*

$A\left(-\frac{x-y}{2}\right) = -\frac{1}{2}\left[A(x) - A(y)\right]$, or

(1.4) $$2A(-\frac{x-y}{2}) = -[A(x)-A(y)]$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called *alternative Jensen type mapping* if A satisfies the functional equation (1.4).

Banach Contraction Principle 1.1 ([4]). *If* (X, d) *is a complete metric space and* $J: X \to X$ *is a mapping such that* $d(Jx, Jy) \le L\, d(x, y)$ *, for all* $x, y \in X$ *and a Lipschitz constant* $L < 1$*,then* J *has one, and only one, fixed point* $x^* = J(x^*) = Jx^*$*, such that* $\lim_{n\to\infty} J^n x = x^*$ *for any starting point* $x \in X$*, as well as* $d(J^n x, x^*) \le L^n\, d(x, x^*)$*, for all* $n \ge 0$ *and* $x \in X$ *;*

$d(J^n x, x^*) \le \frac{1}{1-L} d(J^n x, J^{n+1}x)$*, for all* $n \ge 0$ *and* $x \in X$ *; and* $d(x, x^*) \le \frac{1}{1-L} d(x, Jx)$*,*

for all $x \in X$.

We note that $d(Jx, x^*) = d(Jx, Jx^*) \le L\, d(x, x^*)$. Thus

$d(J^2x, x^*) = d(J^2x, Jx^*) = d(J^2x, J^2x^*) = d(J\, Jx, J\, Jx^*) \le L\, d(Jx, Jx^*) \le L^2\, d(x, x^*)$.

Thus by induction on $n \in \mathbb{N}$ we get $d(J^n x, x^*) \le L^n\, d(x, x^*)$.

Alternative Contraction Principle 1.2 ([4], [10]). *If* (X, d) *is a complete metric space and* $J: X \to X$ *is a strictly contractive mapping (that is :* $d(Jx, Jy) \le L\, d(x, y)$ *, for all* $x, y \in X$ *and a Lipschitz constant* L (<1) *), then either* $d(J^n x, J^{n+1}x) = \infty$*, for all* $n \ge 0$*, or there exists* $n_0 \in N = \{1, 2, \ldots\}$ *such that* $d(J^n x, J^{n+1}x) < \infty$*, for all* $n \ge n_0$ *; the sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J *;* y^* *is the unique fixed point of* J *in the set*

$Y = \{y \in X, d(J^{n_0}x, y) < \infty\}$*; and* $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$*, for all* $y \in Y$.

2. STABILITY OF THE ALTERNATIVE JENSEN AND JENSEN TYPE MAPPINGS

Throughout this paper, let E be a linear space and F a Banach space.

Theorem 2.0. (Cadariu - Radu [1]). *If $f: E \to F$ satisfies $f(0) = 0$ and the approximately Jensen inequality*

$$\left\| 2f(\frac{x+y}{2}) - f(x) - f(y) \right\| \le \delta$$

for some fixed $\delta \ge 0$ and all $x, y \in E$, then there exists a unique additive mapping $j: E \to F$ satisfying the inequality

$$\| f(x) - j(x) \| \le \delta \left(= \frac{L}{1 - L} \delta \ \ for \ \ L = \frac{1}{2} \right)$$

for all $x \in E$.

We prove the following Theorem on alternative Jensen mappings via the alternative contraction principle.

Theorem 2.1. *If $f: E \to F$ satisfies the approximately alternative Jensen inequality*

(2.1) $$\left\| 2f\left(-\frac{x_1 + x_2}{2} \right) + f(x_1) + f(x_2) \right\| \le \delta$$

for some fixed $\delta \ge 0$ and all $x_1, x_2 \in E$, then there exists a unique alternative Jensen mapping j: $E \to F$ satisfying $\|f(0)\| \le \delta / 4$ and inequality

(2.2) $$\|f(x) - j(x)\| \le \frac{L}{1-L}(\frac{3}{2}\delta + \|f(0)\|)(\le \frac{7}{4}\delta) \ \ for \ L = \frac{1}{2}(<1),$$

and all $x \in E$.

Proof. Let us denote $\delta_{1.5} = 1.5\delta + \|f(0)\|$ and $X = \{ g : E \to F \}$.Then we introduce *the metric*

(2.3) $$d(g,h) = \inf\left\{ \varepsilon \in \mathbb{R}^+ = (0,\infty), \ \|g(x) - h(x)\| \le \delta_{1.5}\varepsilon, \ \forall x \in E \right\}$$
$$(\langle \ \varepsilon)$$

for all g, h $\in$ X.

It is clear that (X, d) is *complete.*

Consider the linear mapping $J: X \to X$, such that $Jg(x) = 2^{-1} g(2x)$; thus,

$$J^n g(x) = 2^{-n} g(2^n x).$$

Replacing $2x$ on x in inequality $\| g(x) - h(x) \| \le \delta_{1.5}\varepsilon$ of the above metric (2.3), and then dividing by 2, one gets

$$\|Jg(x) - Jh(x)\| = 2^{-1}\|g(2x) - h(2x)\| \le 2^{-1}\delta_{1.5}\varepsilon \ ,$$

or $d(Jg,Jh)\le L\varepsilon$ (for L=1/2), yielding

$$d(Jg,Jh)\le L\,d(g,h),$$

for all $g, h \in X$. Thus $J{:}X\to X$ is *strictly contractive* with Lipschitz constant L=1/2.

Setting $x_1=x_2=0$ in (2.1) we get $\|f(0)\|\le\delta/4$ .Besides placing $x_1=x_2=x$ in (2.1)

We find $\|f(-x)+f(x)\|\le\delta/2$ .Thus setting x_1 = -2x, x_2 = 0 in (2.1) and employing the triangle inequality, we obtain $\|\,2f(x)+f(-2x)+f(0)\,\|\le\delta$, for all $x\in X$.

Therefore from these inequalities and the triangle inequality, we get

$$\|2f(x)-f(2x)\|\le\|f(-2x)+f(0)+2f(x)\|+\|-[f(-2x)+f(2x)]\|+\|-[f(0)]\|$$
$$\le\frac{3}{2}\delta+\|f(0)\|(\le\frac{7\delta}{4}),\quad or$$

$$\|f(x)-Jf(x)\,\|=\|f(x)-2^{-1}f(2x)\,\|\le\left(\frac{3}{2}\delta+\|f(0)\,\|\,\right)\Big/2=\delta_{1.5}/2=\delta_{1.5}L\,.$$

Thus $d(f,Jf)\le L$.

Employing *the alternative contraction principle 1.2*, one establishes *the existence of* an alternative Jensen mapping $j:X\to X$ such that $j(x)=2^{-1}j(2x)$; that is, $j:X\to X$ is *a fixed point of* $J:X\to X$,such that $Jj(x)=j(x)=2^{-1}j(2x)$ in $Y=\{\,g\in X,d(f,g)\langle\infty\,\}$.

Thus j: $X\to X$ is *the unique mapping* satisfying $j(x)=2^{-1}j(2x)$ and inequality

$$\|\,f(x)-j(x)\,\|\le\delta_{1.5}\varepsilon\quad(=\delta_{1.5}\;\;if\;\;\varepsilon=1),$$

for all $x\in E$.

Besides

$$0\le d(J^n f,j)\le L^n d(f,j)=2^{-n}\varepsilon\to 0,\text{ as } n\to\infty,\text{ or }\lim_{n\to\infty}d(J^n f,j)=0\,.$$

This implies

(2.4) $$\lim_{n\to\infty}J^n f(x)=\lim_{n\to\infty}2^{-n}f(2^n x)=j(x)\;,$$

for all $x\in E$.

We note that $Jf(x)=2^{-1}f(2x)$ yields

$$J^2 f(x)=J(Jf(x))=J(2^{-1}f(2x))=2^{-1}2^{-1}f(2(2x))=2^{-2}f(2^2x).$$

Thus by induction on $n\in\mathbb{N}$, one gets $J^n f(x)=2^{-n}f(2^n x),\;\;\forall n\in\mathbb{N}$.

Therefore from the triangle inequality, we find

$$\|f(x)-J^n f(x)\| \ \left(=\|f(x)-2^{-n} f(2^n x)\|\right)$$
$$\leq \|f(x)-Jf(x)\|$$
$$+\|Jf(x)-J^2 f(x)\|+\ldots+\|J^{n-1} f(x)-J^n f(x)\|$$
$$\leq \frac{\delta_{1.5}}{2}\left(1+2^{-1}+\ldots+2^{-(n-1)}\right)$$
$$=\delta_{1.5}\left(1-2^{-n}\right)\to\delta_{1.5}, \ \ as \ \ n\to\infty.$$

Therefore

$$\|f(x)-j(x)\|\leq\|f(x)-J^n f(x)\|+\|J^n f(x)-j(x)\|\to\delta_{1.5}+0=\delta_{1.5}, \quad n\to\infty,$$

satisfying inequality (2.2). Also

$$d(f,j)\leq\frac{1}{1-L}d(f,df)\leq\frac{L}{1-L}=1.$$

Replacing $(2^n x_1, 2^n x_2)$ on (x_1, x_2) in (2.1), and multiplying by 2^{-n}, we get from (2.4) by letting $n\to\infty$, that

$$\left\|2\cdot 2^{-n} f\left(2^n\left(-\frac{x_1+x_2}{2}\right)\right)+2^{-n} f\left(2^n x_1\right)+2^{-n} f\left(2^n x_2\right)\right\|\leq 2^{-n}\delta\to 0, \ as \ \ n\to\infty,$$

or that $j : X\to X$ is *an alternative Jensen mapping*, satisfying

(2.5) $$2j\left(-\frac{x_1+x_2}{2}\right)=-[j(x_1)+j(x_2)],$$

for all $x_1, x_2 \in E$, completing the proof of Theorem 2.1.

Corollary 2.1. *If* $f : E\to F$ *satisfies the approximately alternative Jensen inequality (2.1), then there exists a unique additive mapping* $j : E\to F$ *satisfying inequality (2.2).*

Proof. Following the proof of Theorem 2.1 we get that there exists a unique alternative Jensen mapping $j : E\to F$ satisfying the alternative Jensen equation (2.5).

However, if we replace $(2^{n+1} x_1, 2^{n+1} x_2)$ on (x_1, x_2) in (2.1), and multiply by $2^{-(n+1)}$, we get from (2.4) by letting $n\to\infty$, that

$$\left\|2\cdot 2^{-(n+1)} f\left(2^{n+1}\left(-\frac{x_1+x_2}{2}\right)\right)+2^{-(n+1)} f\left(2^{n+1} x_1\right)+2^{-(n+1)} f\left(2^{n+1} x_2\right)\right\|$$
$$=\left\|2^{-n} f\left(-2^n\left(x_1+x_2\right)\right)+2^{-(n+1)} f\left(2^{n+1} x_1\right)+2^{-(n+1)} f\left(2^{n+1} x_2\right)\right\|$$
$$\leq \ 2^{-(n+1)}\delta\to 0, as \ \ n\to\infty, \text{ or}$$
$$0\leq\|j(-(x_1+x_2))+j(x_1)+j(x_2)\|\leq 0, \text{ or}$$

(2.6) $$j(-(x_1+x_2))=-[j(x_1)+j(x_2)]\ .$$

Now letting $x_1=x_2=0$ in (2.6),we get *j*(0)=0.Thus placing $x_1=x,x_2=0$ in (2.6),one finds *j(-x)=-j(x)*.Therefore, we find from (2.6) that $j(x_1+x_2)=-j(-(x_1+x_2))=j(x_1)+j(x_2)$;

that is, *a unique additive mapping* $j:E\to F$, satisfying (2.2), and thus the proof of this Corollary is complete.

Claim that an alternative Jensen mapping $j:E\to F$ *satisfying (2.5) is additive*.

In fact, letting $x_1=x_2=0$ in (2.5),we get *j*(0)=0.Thus placing $x_1=x,x_2=-x$ in (2.5),we find that $j(-x)=-j(x)$.Therefore, if we take $x_1=-2x$, $x_2=0$ in (2.5),then

$$2\mathrm{j}(x)=-j(-2x)-j(0)=j(2x),\text{ or}$$

(2.7) $$j(2x)=2j(x),$$

for all $x\in X$. Thus replacing $(-2x_1,-2x_2)$ on (x_1, x_2) in (2.5) we find from (2.7) that

$$j(x_1+x_2)=j(x_1)+j(x_2),$$

for all $x_1, x_2\in X$. Therefore $j:E\to F$ is additive.

Similarly we establish the following Theorem 2.2.

Theorem 2.2. *If* $f: E\to F$ *satisfies the approximately alternative Jensen type inequality*

(2.8) $$\left\|2f(-\frac{x_1-x_2}{2})+f(x_1)-f(x_2)\right\|\le\delta$$

for some fixed $\delta\ge 0$ *and all* $x_1, x_2\in E$*, then there exists a unique alternative Jensen type mapping* $j: E\to F$ *satisfying* $\|f(0)\|\le\delta/2$ *and inequality*

(2.9) $$\|f(x)-j(x)\|\le\frac{L}{1-L}(2\delta+\|f(0)\|)\quad \textit{for } L=\frac{1}{2},$$

and all $x\in X$.

Proof. Replacing $x_1=x_2=0$ in inequality (2.8) ,we get $\|f(0)\|\le\delta/2$.Besides, setting $x_1=x,x_2=-x$ in (2.8),one finds $\|f(-x)+f(x)\|\le\delta$.Also substituting $x_1=-2x,x_2=0$ in (2.8),we obtain $\|2f(x)+f(-2x)-f(0)\|\le\delta$.Therefore from these inequalities and the triangle inequality we establish the basic inequality

(2.10) $$\begin{aligned}&\|2f(x)-f(2x)\|\le\|2f(x)+f(-2x)-f(0)\|\\&+\|-[f(-2x)+f(2x)]\|+\|f(0)\|\le 2\delta+\|f(0)\|\end{aligned}$$

We omit the rest of the proof as similar to the proof of the above Theorem 2.1.

Corollary 2.2. *If $f: E \to F$ satisfies the approximately alternative Jensen type inequality (2.8), then there exists a unique additive mapping $j: E \to F$ satisfying inequality (2.9).*

3. STABILITY OF THE ALTERNATIVE ADDITIVE EQUATION (1.1) OF THE FIRST FORM

Theorem 3.1. *If $f: E \to F$ satisfies the approximately alternative additive inequality of the first form*

(3.1) $$\| f(x_1 + x_2) + f(x_1 - x_2) + 2f(-x_1) \| \le \delta$$

for some fixed $\delta \ge 0$ and all $x_1, x_2 \in E$, then there exists a unique alternative additive mapping j: $E \to F$ of the first form satisfying $\|f(0\| \le \delta / 4$ and inequality

(3.2) $$\|f(x) - j(x)\| \le \delta_{1.5} \left(= \frac{L}{1\text{-}L}\delta_{1.5} \text{ for } L = \frac{1}{2} \right),$$

for all $x \in E$.

Proof. In fact, we set $x_1 = x_2 = x$ in (3.1) and $Jf(x) = 2^{-1} f(2x)$, one gets that

(3.3) $$\|f(x) - Jf(x)\| = \|f(x) - 2^{-1} f(2x)\| \le \frac{\delta_{1.5}}{2} = \delta_{1.5} L,$$

for all $x \in E$. Replacing $(2^n x_1, 2^n x_2)$ on (x_1, x_2) in (3.1), dividing by 2^n, and considering $j(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, we get the alternative additive equation of the first form

(3.4) $$j(x_1 + x_2) + j(x_1 - x_2) = -2j(-x_1),$$

for all $x_1, x_2 \in E$. Thus $j: E \to F$ is alternative additive of the first form. The rest of the proof is omitted as similar to the proof of Theorem 2.1.

Corollary 3.1. *If $f: E \to F$ satisfies the approximately alternative additive inequality of the first form (3.1), then there exists a unique alternative additive mapping $j: E \to F$ satisfying inequality (3.2).*

Proof. Following the proof of Theorem 3.1, we get that there exists a unique alternative additive mapping $j: E \to F$ of the first form satisfying equation (3.4).

Claim that an alternative additive mapping $j: E \to F$ of the first form is additive.

In fact, setting $x_1 = x_2 = 0$ *and* $x_1 = 0, x_2 = x$ in equation (3.4), we get that

(3.5) $$j(0) = 0,$$

and

(3.6) $$j(-x) = -j(x),$$

respectively,for all $x \in X$. Besides replacing (x_2, x_1) on (x_1, x_2) in (3.4), one obtains that

(3.7) $$j(x_1 + x_2) - j(x_1 - x_2) = -2j(-x_2)$$

for all $x_1, x_2 \in E$. Thus adding (3.4) and (3.7), we find

(3.8) $$j(x_1 + x_2) = -[j(-x_1) + j(-x_2)] = j(x_1) + j(x_2)$$

for all $x_1, x_2 \in E$, completing the proof of the additivity of $j : E \to F$.

4. STABILITY OF THE ALTERNATIVE ADDITIVE EQUATION (1.4) OF THE SECOND FORM

Theorem 4.1. *If f: $E \to F$ satisfies the approximately alternative additive inequality of the second form*

(4.1) $$\| f(x_1 + x_2) - f(x_1 - x_2) + 2f(-x_2) \| \le \delta$$

for some fixed $\delta \ge 0$ and all $x_1, x_2 \in E$, then there exists a unique alternative additive mapping j: $E \to F$ of the second form satisfying $\| f(0) \| \le \delta / 2$ and inequality

(4.1a) $$\|f(x) - j(x)\| \le \frac{L}{1-L}(2\delta + \|f(0)\|) \ \textit{for}\ L = 1/2,$$

for all $x \in E$.

Proof. In fact, setting $x_1 = x_2 = 0$ in (4.1), one gets

(4.2) $$\|f(0)\| \le \delta / 2$$

for some fixed $\delta \ge 0$.

Similarly, if one substitutes $x_1 = 0, x_2 = -x$ in (4.1), then $\|f(-x) + f(x)\| \le \delta$.

Besides placing $(-x,-x)$ on (x_1, x_2) in (4.1) we find $\| f(-2x) - f(0) + 2f(x) \| \le \delta$.

From these inequalities and the triangle inequality, we get

$$\begin{aligned} &\|2f(x) - f(2x)\| \\ &\le \|f(-2x) - f(0) + 2f(x)\| + \|-[f(-2x) + f(2x)]\| + \|f(0)\| \\ &\le 2\delta + \|f(0)\| (\le 5\delta / 2). \end{aligned}$$

Replacing $(2^n x_1, 2^n x_2)$ on (x_1, x_2) in (4.1), dividing by 2^n,

and considering $j(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$, we get the alternative additive equation of the second form

(4.3) $$j(x_1 + x_2) - j(x_1 - x_2) = -2j(-x_2),$$

for all $x_1, x_2 \in E$. Thus j: $E \rightarrow F$ is alternative additive of the second form. We note that if we set $x_1 = x_2 = 0$ in (4.3), one finds that $j(0) = 0$.The rest of the proof is omitted as similar to the proof of Theorem 2.1.

Corollary 4.1. *If* $f : E \rightarrow F$ *satisfies the approximately alternative additive inequality of the second form (4.1), then there exists a unique alternative additive mapping* j: $E \rightarrow F$ *satisfying inequality (4.1a).*

Proof. Following the proof of Theorem 2.1 we get that there exists a unique alternative additive mapping j: $E \rightarrow F$ of the second form.

It is easy to prove that *an alternative additive mapping* $j : E \rightarrow F$ *of the second form is additive.*

In fact, we follow the proof of Corollary 2.1.

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