Bol. Soc. Mat. Mexicana (3) Vol. 11, 2005

ON THE GENERAL QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. In 1940 and in 1964 S. M. Ulam proposed the general problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?". In 1941 D. H. Hyers solved this stability problem for linear mappings. In 1951 D. G. Bourgin was the second author to treat the same problem for additive mappings. According to P. M. Gruber (1978) this kind of stability problems are of particular interest in probability theory and in the case of functional equations of different types. In 1981 F. Skof was the first author to solve the Ulam problem for quadratic mappings. In 1982–2002 we solved the above Ulam problem for linear and nonlinear mappings and established analogous stability problems even on restricted domains. Further, we applied some of our recent results to the asymptotic behavior of functional equations of different types. The purpose of this paper is the stability result for generalized quadratic mappings.

1. Introduction

In 1940 and in 1964 S. M. Ulam [27] proposed the general problem:

"When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?".

In 1941 D. H. Hyers [13] solved this stability problem for linear mappings. In 1951 D. G. Bourgin [3] was the second author to treat the same problem for additive mappings. According to P. M. Gruber [12] (1978), this kind of stability problems are of particular interest in *probability theory* and in the case of *functional equations* of different types. In 1978 Th. M. Rassias [22] employed Hyers' ideas to new additive mappings. In 1981 and 1983 F. Skof [23], [24] was the first author to solve the Ulam problem for quadratic mappings. In 1982-2002 we ([16], [17], [18], [19], [20], [21]) solved the above Ulam problem for linear and nonlinear mappings and established analogous stability problems on restricted domains (see also [14]). Further, we applied some of our recent results to the asymptotic behavior of functional equations of different types. In 1999 P. Gavruta [11] answered a question of ours [16] concerning the stability of Cauchy equation. In 1996 and 1998 we [19], [20] solved the Ulam stability problem for quadratic mappings $Q : X \to Y$ satisfying the functional equation

$$Q(a_1x_1 + a_2x_2) + Q(a_2x_1 - a_1x_2) = (a_1^2 + a_2^2)[Q(x_1) + Q(x_2)]$$

for every $x_1, x_2 \in X$, and fixed reals $a_1, a_2 \neq 0$, where X and Y are real normed linear spaces. The purpose of this paper is the stability result for generalized

2000 *Mathematics Subject Classification*: 39B.
Keywords and phrases: Ulam problem, stability, general quadratic mapping.

quadratic mappings $Q : X \to Y$ satisfying $Q(0) = 0$ and the following quadratic functional equation

$$(*) \qquad Q\left(\sum_{i=1}^{p} a_i x_i\right) + \sum_{1 \le i < j \le p} Q(a_j x_i - a_i x_j) = m \sum_{i=1}^{p} Q(x_i)$$

for every $x_i \in X (i = 1, 2, \ldots, p)$, and fixed $a_i \neq 0$ $(i = 1, 2, \ldots, p)$, $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, p)$, where $\mathbb{R} :=$ set of reals and p is arbitrary but fixed and equals to $2, 3, 4, \ldots,$ such that $0 < m = \sum\limits_{i=1}^{p} a_i^2$.

If X and Y are normed linear spaces and Y is complete, then we establish an approximation of approximately quadratic mappings $f : X \to Y$ by quadratic mappings $Q : X \to Y$, such that $f(0) = 0$ and the corresponding approximately quadratic functional inequality

$$(**) \quad \left\|f\left(\sum_{i=1}^{p} a_i x_i\right) + \sum_{1 \le i < j \le p} f(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(x_i)\right]\right\| \le c \prod_{i=1}^{p} ||x_i||^{r_i}$$

holds with constants $c \ge 0$ (independent of $x_i \in X : i = 1, 2, \ldots, p$), and any fixed reals a_i and $r_i > 0 (i = 1, 2, \ldots, p)$. Denote

$$I_1 = \{(r, m) \in \mathbb{R}^2 : 0 < r < 2, \ m > 1 \quad \text{or} \quad r > 2, \ 0 < m < 1\},$$
$$I_2 = \{(r, m) \in \mathbb{R}^2 : 0 < r < 2, \ 0 < m < 1 \quad \text{or} \quad r > 2, \ m > 1\},$$
$$I_3 = \{(r, m) \in \mathbb{R}^2 : 0 < r < 2, \ m = 1 = pb^2, \ a_i = b = p^{-1/2} : i = 1, 2, \ldots, p\},$$
$$I_4 = \{(r, m) \in \mathbb{R}^2 : r > 2, \ m = 1 = pb^2, \ a_i = b = p^{-1/2} : i = 2, \ldots, p\},$$

where $r = \sum\limits_{i=1}^{p} r_i > 0$, where p is arbitrary but fixed and equals to $2, 3, 4, \ldots$. Note that $m^{r-2} < 1$ if $(r, m) \in I_1$, $m^{2-r} < 1$ if $(r, m) \in I_2$, $p^{r-2} < 1$ if $(r, m) \in I_3$, and $p^{2-r} < 1$ if $(r, m) \in I_4$. Also denote $\gamma = \prod\limits_{i=1}^{p} |a_i|^{r_i} > 0$. Also denote

$$f_n(x) = \begin{cases} m^{-2n} f(m^n x), & \text{if} \quad (r, m) \in I_1 \\ m^{2n} f(m^{-n} x), & \text{if} \quad (r, m) \in I_2 \\ p^{-n} f(p^{n/2} x), & \text{if} \quad (r, m) \in I_3 \\ p^{n} f(p^{-n/2} x), & \text{if} \quad (r, m) \in I_4 \end{cases}$$

for all $x \in X$ and $n \in \mathbb{N} : p = 2, 3, 4, \ldots$.

Definition (1.1). Let X and Y be real normed linear spaces. Let $a = (a_1, a_2, \ldots, a_p) \neq (0, 0, \ldots, 0)$ with $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, p)$. Then a mapping $Q : X \to Y$ is called *quadratic with respect to* $a : |a| = \left(\sum\limits_{i=1}^{p} a_i^2\right)^{1/2}$, if the generalized quadratic functional equation (*) holds for every $x_i \in X (i = 1, 2, \ldots, p)$. Denote

$$(1.2) \qquad \overline{Q}(x) = \begin{cases} \sum\limits_{i=1}^{p} Q(a_i x) / \sum\limits_{i=1}^{p} a_i^2, & \text{if} \quad (r, m) \in I_1, \\ (\sum\limits_{i=1}^{p} a_i^2)\left[\sum\limits_{i=1}^{p} Q\left(a_i x / \sum_{i=1}^{p} a_i^2\right)\right], & \text{if} \quad (r, m) \in I_2, \end{cases}$$

for all $x \in X$.

2. Quadratic functional stability

THEOREM (2.1). *Let X and Y be normed linear spaces. Assume that Y is complete. Assume in addition that the mapping $f : X \to Y$ satisfies $f(0) = 0$ and the approximately quadratic functional inequality (**) for every $x_i \in X$ $(i = 1, 2, \ldots, p)$. If $r \neq 2$ and $p \geq 2$, then the limit*

$$Q(x) = \lim_{n\to\infty} f_n(x) \tag{2.2}$$

exists for all $x \in X$ and $Q : X \to Y$ is the unique quadratic mapping such that

$$||f(x) - Q(x)|| \leq ||x||^r \begin{cases} \gamma c/(m^2 - m^r), & \text{if} \quad (r, m) \in I_1 \\ \gamma c/(m^r - m^2), & \text{if} \quad (r, m) \in I_2 \\ c/(p - p^{r/2}), & \text{if} \quad (r, m) \in I_3 \\ c/(p^{r/2} - p), & \text{if} \quad (r, m) \in I_4 \end{cases} \tag{2.3}$$

holds for all $x \in X$.

Proof. From the hypotheses of this theorem, the following condition

$$f(0) = 0 \tag{2.4}$$

is useful to hold. We claim for each $n \in \mathbb{N}$ that

$$||f(x) - f_n(x)|| \leq ||x||^r \begin{cases} \frac{\gamma c}{m^2 - m^r}(1 - m^{n(r-2)}), & \text{if} \quad (r, m) \in I_1, \\ \frac{\gamma c}{m^r - m^2}(1 - m^{n(2-r)}), & \text{if} \quad (r, m) \in I_2, \\ \frac{c}{p - p^{r/2}}(1 - p^{n(r-2)/2}), & \text{if} \quad (r, m) \in I_3, \\ \frac{c}{p^{r/2} - p}(1 - p^{n(2-r)/2}), & \text{if} \quad (r, m) \in I_4 \end{cases} \tag{2.5}$$

for all $x \in X$. By replacing $Q, \overline{Q}$ of (1.2) with $f, \overline{f}$, respectively, one denotes:

$$\overline{f}(x) = \begin{cases} \sum\limits_{i=1}^{p} f(a_i x) / \sum\limits_{i=1}^{p} a_i^2, & \text{if} \quad (r, m) \in I_1 \\ \\ (\sum\limits_{i=1}^{p} a_i^2) \left[\sum\limits_{i=1}^{p} f\left(a_i x / \sum\limits_{i=1}^{p} a_i^2 \right) \right], & \text{if} \quad (r, m) \in I_2 \end{cases} \tag{2.6}$$

holds for all $x \in X$. From (2.4), (2.6) and (**), with $x_i = a_i x$ $(i = 1, 2, \ldots, p)$, we obtain

$$\left\| f(mx) + \binom{p}{2} f(0) - m \sum_{i=1}^{p} f(a_i x) \right\| \leq \gamma c \|x\|^r, \quad \text{or}$$

$$\left|\left| f(mx) - m \left[\sum_{i=1}^{p} f(a_i x) \right] \right|\right| \leq \gamma c ||x||^r, \quad \text{or}$$

$$||m^{-2} f(mx) - \overline{f}(x)|| \leq \frac{\gamma c}{m^2} ||x||^r, \tag{2.7}$$

if I_1 holds. Besides from (2.4), (2.6) and (**), with $x_1 = x, x_j = 0$ $(j = 2, 3, \ldots, p)$, we get

$$\left|\left| f(a_1 x) + \sum_{j=2}^{p} f(a_j x) - m[f(x) + (p-1) f(0)] \right|\right| \leq 0,$$

$$\text{or} \quad \left\| \sum_{i=1}^{p} f(a_i x) - mf(x) \right\| \leq 0, \quad \text{or}$$

$$\overline{f}(x) = f(x), \tag{2.8}$$

if I_1 holds. Therefore from (2) and (2.8) we have

$$||f(x) - m^{-2} f(mx)|| \leq \frac{\gamma c}{m^2} ||x||^r = \frac{\gamma c}{m^2 - m^r}(1 - m^{r-2}) ||x||^r, \tag{2.9}$$

which is (2.5) for $n = 1$, if I_1 holds.

Similarly, from (2.4), (2.6) and (**), with $x_i = \frac{a_i}{m} x (i = 1, 2, \ldots, p)$, we obtain

$$\left\| f(x) + \binom{p}{2} f(0) - m \sum_{i=1}^{p} f\left(\frac{a_i}{m} x\right) \right\| \leq \frac{\gamma c}{m^r} ||x||^r, \quad \text{or}$$

$$||f(x) - \overline{f}(x)|| \leq \frac{\gamma c}{m^r} ||x||^r, \tag{2.10}$$

if I_2 holds. Further from (2.4), (2.6) and (**), with $x_1 = \frac{x}{m}, x_j = 0$ $(j = 2, 3, \ldots, p)$, we get

$$\left\| f\left(\frac{a_1}{m} x\right) + \sum_{j=2}^{p} f\left(\frac{a_j}{m} x\right) - m[f(m^{-1} x) + (p-1) f(0)] \right\| \leq 0, \quad \text{or}$$

$$\left\| \sum_{i=1}^{p} f\left(\frac{a_i}{m} x\right) - mf(m^{-1} x) \right\| \leq 0, \quad \text{or}$$

$$\overline{f}(x) = m^2 f(m^{-1} x), \tag{2.11}$$

if I_2 holds. Therefore from (2.10) and (2.11) we have

$$||f(x) - m^2 f(m^{-1} x)|| \leq \frac{\gamma c}{m^r} ||x||^r = \frac{\gamma c}{m^r - m^2}(1 - m^{2-r}) ||x||^r, \tag{2.12}$$

which is (2.5) for $n = 1$, if I_2 holds.

Also, with $x_i = x$ $(i = 1, 2, \ldots, p)$ in (**) and $a_i = b = p^{-1/2}$ $(i = 1, 2, \ldots, p)$, we obtain

$$\left\| f(p^{1/2} x) - pf(x) \right\| \leq c ||x||^r, \quad \text{or}$$

$$\left\| f(x) - p^{-1} f(p^{1/2} x) \right\| \leq \frac{c}{p} \|x\|^r = \frac{c}{p - p^{r/2}} \left[1 - p^{(r-2)/2}\right] \|x\|^r, \tag{2.13}$$

which is (2.5) for $n = 1$, if I_3 holds.

In addition, with $x_i = p^{-1/2} x$ $(i = 1, 2, \ldots, p)$ in (**) and $a_i = b = p^{-1/2}$ $(i = 1, 2, \ldots, p)$, we obtain

$$\left\| f(x) - pf(p^{-1/2} x) \right\| \leq c p^{-r/2} ||x||^r, \quad \text{or}$$

$$\left\| f(x) - pf(p^{-1/2} x) \right\| \leq c p^{-r/2} ||x||^r = \tfrac{c}{p^{r/2} - p}[1 - p^{(2-r)/2}] ||x||^r, \tag{2.13a}$$

which is (2.5) for $n = 1$, if I_4 holds.

Assume (2.5) is true if $(r, m) \in I_1$. From (2.9), with $m^n x$ in place of x, and from the triangle inequality, we have

$$||f(x) - f_{n+1}(x)|| = ||f(x) - m^{-2(n+1)} f(m^{n+1}x)||$$

$$\begin{aligned} &\leq ||f(x) - m^{-2n} f(m^n x)|| + ||m^{-2n} f(m^n x) - m^{-2(n+1)} f(m^{n+1}x)|| \\ &\leq \frac{\gamma c}{m^2 - m^r}[(1 - m^{n(r-2)}) + m^{-2n}(1 - m^{r-2})m^{nr}]||x||^r \\ &= \frac{\gamma c}{m^2 - m^r}(1 - m^{(n+1)(r-2)})||x||^r, \end{aligned} \tag{2.14}$$

if I_1 holds.

Similarly assume (2.5) is true if $(r, m) \in I_2$. From (2.12), with $m^{-n}x$ in place of x, and the triangle inequality, we have

$$||f(x) - f_{n+1}(x)|| = ||f(x) - m^{2(n+1)} f(m^{-(n+1)}x)||$$

$$\begin{aligned} &\leq ||f(x) - m^{2n} f(m^{-n} x)|| + ||m^{2n} f(m^{-n} x) - m^{2(n+1)} f(m^{-(n+1)}x)|| \\ &\leq \frac{\gamma c}{m^r - m^2}[(1 - m^{n(2-r)}) + m^{2n}(1 - m^{2-r})m^{-nr}]||x||^r \\ &= \frac{\gamma c}{m^r - m^2}(1 - m^{(n+1)(2-r)})||x||^r, \end{aligned} \tag{2.15}$$

if I_2 holds.

Also, assume (2.5) is true if $(r, m) \in I_3$. From (2.13), with $(pb)^n x$ $(= p^{n/2}x)$ in place of x, and the triangle inequality, we have

$$||f(x) - f_{n+1}(x)|| = \left|\left|f(x) - p^{-(n+1)} f\left(p^{\frac{n+1}{2}}x\right)\right|\right| = ||f(x) - p^{-(n+1)} f((pb)^{n+1}x)||$$

$$\begin{aligned} &\leq ||f(x) - p^{-n} f((pb)^n x)|| + ||p^{-n} f((pb)^n x) - p^{-(n+1)} f((pb)^{n+1}x)|| \\ &\leq \frac{c}{p - p^{r/2}}\{[1 - p^{n(r-2)/2}] + p^{-n}[1 - p^{(r-2)/2}](pb)^{nr}\}||x||^r \\ &= \frac{c}{p - p^{r/2}}[1 - p^{(n+1)(r-2)/2}]||x||^r, \end{aligned} \tag{2.16}$$

if I_3 holds.

In addition, assume (2.5) is true if $(r, m) \in I_4$. From (2.13a), with $(pb)^n x$ $(= p^{n/2}x)$ in place of x, and the triangle inequality, we have

$$||f(x) - f_{n+1}(x)|| = \left|\left|f(x) - p^{n+1} f\left(p^{\frac{n+1}{2}}x\right)\right|\right| = ||f(x) - p^{n+1} f((pb)^{-(n+1)}x)||$$

$$\begin{aligned} &\leq ||f(x) - p^n f((pb)^{-n} x)|| + ||p^n f((pb)^{-n} x) - p^{n+1} f((pb)^{-(n+1)}x)|| \\ &\leq \frac{c}{p^{r/2} - p}\{[1 - p^{n(2-r)/2}] + p^n[1 - p^{(2-r)/2}](pb)^{-nr}\}||x||^r \\ &= \frac{c}{p^{r/2} - p}[1 - p^{(n+1)(2-r)/2}]||x||^r, \end{aligned} \tag{2.16a}$$

if I_4 holds.

Therefore inequalities (2.14), (2.15) and (2.16) and (2.16a) prove inequality (2.5) for any $n \in \mathbb{N}$.

We claim now that the sequence $\{f_n(x)\}$ *converges*. To do this it suffices to prove that it is a Cauchy sequence. Inequality (2.5) is involved if $(r, m) \in I_1$. In fact, if $i > j > 0$ and $h_1 = m^j x$, we have:

$$||f_i(x)-f_j(x)||=||m^{-2i}f(m^i x)-m^{-2j}f(m^j x)||=m^{-2j}||m^{-2(i-j)}f(m^{i-j}h_1)-f(h_1)||$$

$$\leq \quad m^{-2j}\frac{\gamma c}{m^2-m^r}(1-m^{(i-j)(r-2)})||h_1||^r < \frac{\gamma c}{m^2-m^r}m^{-2j}||h_1||^r \underset{j\to\infty}{\longrightarrow} 0, \tag{2.17}$$

if I_1 holds: $m^{r-2} < 1$.

Similarly, if $h_2 = m^{-j}x$ in I_2, we have:

$$||f_i(x)-f_j(x)||=||m^{2i}f(m^{-i}x)-m^{2j}f(m^{-j}x)||=m^{2j}||m^{2(i-j)}f(m^{-(i-j)}h_2)-f(h_2)||$$

$$\leq \quad m^{2j}\frac{\gamma c}{m^r-m^2}(1-m^{(i-j)(2-r)})||h_2||^r < \frac{\gamma c}{m^r-m^2}m^{2j}||h_2||^r \underset{j\to\infty}{\longrightarrow} 0, \tag{2.18}$$

if I_2 holds: $m^{2-r} < 1$.

Also, if $h_3 = p^{j/2}x$ in I_3, we have:

$$\begin{aligned}\|f_i(x) - f_j(x)\| &= \|p^{-i}f(p^{i/2}x) - p^{-j}f(p^{j/2}x)\| \\ &= p^{-j}\|p^{-(i-j)}f(p^{(i-j)/2}h_3) - f(h_3)\| \\ &\leq p^{-j}\frac{c}{p-p^{r/2}}(1-p^{(i-j)(r-2)/2})\|h_3\|^r < \frac{c}{p-p^{r/2}}p^{-j}\|h_3\|^r \underset{j\to\infty}{\longrightarrow} 0, \end{aligned} \tag{2.19}$$

if I_3 holds: $p^{r-2} < 1$.

In addition, if $h_4 = p^{-j/2}x$ in I_4, we have:

$$\begin{aligned}||f_i(x) - f_j(x)|| &= ||p^i f(p^{-i/2}x) - p^j f(p^{-j/2}x)|| \\ &= p^j||p^{i-j}f(p^{-(i-j)/2}h_4) - f(h_4)||\end{aligned}$$

$$\leq p^j\frac{c}{p^{r/2}-p}(1-p^{(i-j)(2-r)/2})||h_4||^r < \frac{c}{p^{r/2}-p}p^j||h_4||^r \underset{j\to\infty}{\longrightarrow} 0\,, \tag{2.19a}$$

if I_4 holds: $p^{2-r} < 1$.

Then inequalities (2.17), (2.18) and (2.19) and (2.19a) *define a mapping* $Q : X \to Y$ in p variables $x_i \in X$ $(i = 1, 2, \ldots, p)$, given by (2.2).

Claim that from (**) and (2.2) we can get (*), or equivalently that the aforementioned *well–defined mapping* $Q : X \to Y$ *is quadratic with respect to a* $(\neq 0)$. In fact, it is clear from the functional inequality (**) and the limit (2.2) for $(r, m) \in I_1$ that the following functional inequality

$$\begin{aligned} m^{-2n}\Big|\Big|f\Big(\sum_{i=1}^{p} a_i m^n x_i\Big) \;+\; \sum_{1\leq i<j\leq p} f(a_j m^n x_i - a_i m^n x_j) - \Big(\sum_{i=1}^{p} a_i^2\Big)\Big[\sum_{i=1}^{p} f(m^n x_i)\Big]\Big|\Big| \\ \leq\; m^{-2n} c\prod_{i=1}^{p}||m^n x_i||^{r_i}, \end{aligned}$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, and all $n \in \mathbb{N}$ with $p = 2, 3, 4, \ldots$ and $f_n(x) = m^{-2n}f(m^n x) : I_1$ holds. Therefore

$$\Big|\Big|\lim_{n\to\infty} f_n\Big(\sum_{i=1}^{p} a_i x_i\Big) + \lim_{n\to\infty}\sum_{1\leq i<j\leq p} f_n(a_j x_i - a_i x_j) - \Big(\sum_{i=1}^{p} a_i^2\Big)\Big[\sum_{i=1}^{p}\lim_{n\to\infty} f_n(x_i)\Big]\Big|\Big|$$

$$\leq (\lim_{n\to\infty} m^{n(r-2)})c\prod_{i=1}^{p}||x_i||^{r_i} = 0,$$

because $m^{r-2} < 1$ or

$$\left|\left|Q\left(\sum_{i=1}^{p} a_i x_i\right) + \sum_{1\leq i<j\leq p} Q(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} Q(x_i)\right]\right|\right| = 0, \tag{2.20}$$

i.e., the mapping Q satisfies the quadratic functional equation (*).

Similarly, from (**) and (2.2) for $(r, m) \in I_2$ we get that

$$m^{2n}\left|\left|f\left(\sum_{i=1}^{p} a_i m^{-n} x_i\right) + \sum_{1\leq i<j\leq p} f(a_j m^{-n} x_i - a_j m^{-n} x_j) = \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(m^{-n} x_i)\right]\right|\right|$$

$$\leq m^{2n} c\prod_{i=1}^{p}||m^{-n}x_i||^{r_i},$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, and all $n \in \mathbb{N}$ with $f_n(x) = m^{2n} f(m^{-n}x)$: I_2 holds. Thus

$$\left|\left|\lim_{n\to\infty} f_n\left(\sum_{i=1}^{p} a_i x_i\right) + \lim_{n\to\infty}\sum_{1\leq i<j\leq p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p}\lim_{n\to\infty} f_n(x_i)\right]\right|\right|$$

$$\leq (\lim_{n\to\infty} m^{n(2-r)})c\prod_{i=1}^{p}||x_i||^{r_i} = 0,$$

because $m^{2-r} < 1$, i.e., (2.20) holds and the mapping Q satisfies (*).

Also, from (**) and (2.2) for $(r, m) \in I_3$ we obtain that

$$p^{-n}\left|\left|f\left(\sum_{i=1}^{p} a_i p^{n/2} x_i\right) + \sum_{1\leq i<j\leq p} f(a_i p^{n/2} x_j - a_j p^{n/2} x_i) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(p^{n/2} x_i)\right]\right|\right|$$

$$\leq p^{-n} c\prod_{i=1}^{p}||p^{n/2}x_i||^{r_i},$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, and all $n \in \mathbb{N}$ with $f_n(x) = p^{-n} f(p^{n/2}x)$: I_3 holds. Hence

$$\left|\left|\lim_{n\to\infty} f_n\left(\sum_{i=1}^{p} a_i x_i\right) + \lim_{n\to\infty}\sum_{1\leq i<j\leq p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p}\lim_{n\to\infty} f_n(x_i)\right]\right|\right|$$

$$\leq (\lim_{n\to\infty} p^{n(r-2)/2})c\prod_{i=1}^{p}||x_i||^{r_i} = 0,$$

because $p^{r-2} < 1$, i.e., (2.20) holds and the mapping Q satisfies (*).

In addition, from (**) and (2.2) for $(r, m) \in I_4$ we obtain that

$$p^{n}\left|\left|f\left(\sum_{i=1}^{p} a_i p^{-n/2} x_i\right) + \sum_{1\leq i<j\leq p} f(a_j p^{-n/2} x_i - a_i p^{-n/2} x_j)\right.\right.$$

$$\left.\left. - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(p^{-n/2} x_i)\right]\right|\right| \leq p^{n} c\prod_{i=1}^{p}||p^{-n/2}x_i||^{r_i},$$

holds for all vectors $(x_1, x_2, \dots, x_p) \in X^p$, and all $n \in \mathbb{N}$ with $f_n(x) = p^n f(p^{-n/2}x)$: I_4 holds. Hence

$$\left|\left|\lim_{n\to\infty} f_n\left(\sum_{i=1}^{p} a_i x_i\right) + \lim_{n\to\infty} \sum_{1\le i<j\le p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} \lim_{n\to\infty} f_n(x_i)\right]\right|\right|$$

$$\le (\lim_{n\to\infty} p^{n(2-r)/2})c \prod_{i=1}^{p} ||x_i||^{r_i} = 0,$$

because $p^{2-r} < 1$, i.e., (2.20) holds and the mapping Q satisfies (*).

Therefore (2.20) holds if I_j $(j = 1, 2, 3, 4)$ hold or the mapping Q satisfies the quadratic functional equation (*), completing the proof that Q *is a quadratic mapping with respect to* a in X. It is now clear from (2.5) with $n \to \infty$, as well as from the formula (2.2) that the functional inequality (2.3) holds in X. This completes *the existence proof* of the afore-mentioned Theorem (2.1).

It remains to prove *the uniqueness: Let* $Q' : X \to Y$ *be a quadratic mapping with respect to a satisfying* (2.3), *as well as* Q. *Then* $Q' = Q$.

In fact, the condition

$$Q(x) = \begin{cases} m^{-2n}Q(m^n x), & \text{if} \quad (r, m) \in I_1, \\ m^{2n}Q(m^{-n}x), & \text{if} \quad (r, m) \in I_2, \\ p^{-n}Q(p^{n/2}x), & \text{if} \quad (r, m) \in I_3, \\ p^n Q(p^{-n/2}x), & \text{if} \quad (r, m) \in I_4 \end{cases} \tag{2.21}$$

holds for all $x \in X$ and $n \in \mathbb{N}$ where p is arbitrary but fixed and equals $2, 3, 4, \dots$, as a consequence of (2.5) with $c = 0$. Remember Q' satisfies (2.21) as well for $(r, m) \in I_1$. Then for every $x \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned} ||Q(x) - Q'(x)|| &= ||m^{-2n}Q(m^n x) - m^{-2n}Q'(m^n x)|| \\ &\le m^{-2n}\left\{||Q(m^n x) - f(m^n x)|| + ||Q'(m^n x) - f(m^n x)||\right\} \\ &\le m^{-2n}\frac{2\gamma c}{m^2 - m^r}||m^n x||^r = m^{n(r-2)}\frac{2\gamma c}{m^2 - m^r}||x||^r \to 0 \\ &\quad \text{as } n \to \infty, \end{aligned} \tag{2.22}$$

if I_1 holds: $m^{r-2} < 1$.

Similarly for $(r, m) \in I_2$, we establish

$$\begin{aligned} ||Q(x) - Q'(x)|| &= ||m^{2n}Q(m^{-n}x) - m^{2n}Q'(m^{-n}x)|| \\ &\le m^{2n}\left\{||Q(m^{-n}x) - f(m^{-n}x)|| + ||Q'(m^{-n}x) - f(m^{-n}x)||\right\} \\ &\le m^{2n}\frac{2\gamma c}{m^r - m^2}||m^{-n}x||^r = m^{n(2-r)}\frac{2\gamma c}{m^r - m^2}||x||^r \to 0, \\ &\quad \text{as } n \to \infty, \end{aligned} \tag{2.23}$$

if I_2 holds: $m^{2-r} < 1$.

Also for $(r, m) \in I_3$, we get

$$\begin{aligned} ||Q(x) - Q'(x)|| &= ||p^{-n}Q(p^{n/2}x) - p^{-n}Q'(p^{n/2}x)|| \\ &\leq p^{-n}\left\{||Q(p^{n/2}x) - f(p^{n/2}x)|| + ||Q'(p^{n/2}x) - f(p^{n/2}x)||\right\} \\ &\leq p^{-n}\frac{2c}{p - p^{r/2}}||p^{n/2}x||^r = p^{n(r-2)/2}\frac{2c}{p - p^{r/2}}||x||^r \to 0, \\ &\quad \text{as } n \to \infty, \end{aligned} \tag{2.24}$$

if I_3 holds: $p^{r-2} < 1$.

In addition, for $(r, m) \in I_4$, we get

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \|p^n Q(p^{-n/2}x) - p^n Q'(p^{-n/2}x)\| \\ &\leq p^n\left\{\|Q(p^{-n/2}x) - f(p^{-n/2}x)\| + \|Q'(p^{-n/2}x) - f(p^{-n/2}x)\|\right\} \\ &\leq p^n\frac{2c}{p^{r/2} - p}\|p^{-n/2}x\|^r = p^{n(2-r)/2}\frac{2c}{p^{r/2} - p}\|x\|^r \to 0, \\ &\quad \text{as } n \to \infty, \end{aligned} \tag{2.25}$$

if I_4 holds: $p^{2-r} < 1$.

Thus from (2.22), (2.23), (2.24) and (2.25) we find $Q(x) = Q'(x)$ for all $x \in X$.

This completes the proof of the *uniqueness* and *stability* of the quadratic functional equation (*). □

Acknowledgement

We are grateful to the anonymous referees for their valuable comments.

Received November 24, 2004

Final version received June 14, 2005

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