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ALTERNATIVE CONTRACTION PRINCIPLE AND ULAM STABILITY PROBLEM

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ABSTRACT. In 1964 S. M. Ulam [6] proposed the general Ulam stability problem: *" When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true ?"* Almost all proofs on the Ulam stability of additive functional equations have employed the well-known Hyers (direct) method established by D. H. Hyers [1] in 1941. On the other hand, V. Radu [2] in 2003 noticed that *a fixed point alternative method* is very important for the solution of the Ulam problem. In 2003-2004 M. J. Rassias and the author [3-5] of this paper investigated the Ulam stability of Jensen and Jensen type mappings by applying the Hyers method. In this paper we establish the Ulam stability of Jensen and Jensen type mappings as well as additive mappings of two forms, via *an alternative contraction principle.* This kind of stability problems can be applied in stochastic analysis, actuarial and financial mathematics.

Key words and phrases : Fixed point, Alternative contraction principle, Jensen and Jensen type, Additive mapping of two forms, Hyers (direct) method, Ulam stability.

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1. INTRODUCTION

In 1940 and 1964 S. M. Ulam [6] proposed the general Ulam stability problem:

" When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true ?"

Almost all proofs on the Ulam stability of additive functional equations have employed the well-known *Hyers (direct) method* established by D. H. Hyers [1] in 1941. On the other hand, V. Radu [2] in 2003 noticed that *a fixed point alternative method* is very important for the solution of the Ulam problem. In 2003-2004 M. J. Rassias and the author [3-5] of this paper investigated the Ulam stability of Jensen and Jensen type mappings by applying the Hyers method. In this paper we establish the Ulam stability of Jensen and Jensen type mappings as well as additive mappings of two forms, via *an alternative contraction principle.* This kind of stability problems can be applied in stochastic analysis, actuarial and financial mathematics.

Let X and Y be real linear spaces and $\mathbb{N}$ the set of natural numbers..

Definition 1.1. A mapping $A : X \to Y$ is called *additive of the first form* if A satisfies the functional equation

(1.1) $$A(x_1 + x_2) + A(x_1 - x_2) = 2A(x_1)$$

for all $x_1, x_2 \in X$ with initial condition

(1.2) $$A(0) = 0 .$$

We note that (1.1) is equivalent to *the Jensen equation*

(1.3) $$2A\left(\frac{x+y}{2}\right) = A(x) + A(y)$$

for $x = x_1 + x_2$, $y = x_1 - x_2$.

Definition 1.2. A mapping $A : X \to Y$ is called *additive of the second form* if A satisfies the functional equation

(1.4) $$A(x_1 + x_2) - A(x_1 - x_2) = 2A(x_2)$$

for all $x_1, x_2 \in X$.

We note that (1.4) is equivalent to *the Jensen type equation*

(1.5) $$2A\left(\frac{x-y}{2}\right) = A(x) - A(y)$$

for $x = x_1 + x_2$, $y = x_1 - x_2$.

Definition 1.3. A mapping $f : X \to Y$ is called *approximately odd* if f satisfies the functional inequality

(1.6) $$\| f(-x) + f(x) \| \leq \theta$$

for some fixed $\theta \geq 0$ and for all $x \in X$.

We note that if $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, then from (1.6) one gets that

$$0 \leq \| A(-x) + A(x) \| = \lim_{n\to\infty} 2^{-n} \| f(-2^n x) + f(2^n x) \| \leq \lim_{n\to\infty} 2^{-n}\theta = 0 \text{ , or}$$

$$A(-x) = -A(x) \text{ , for all } x \in X.$$

Banach Contraction Principle 1.1 ([2]). *If* (X, d) *is a complete metric space and* J:$X\to X$ *a mapping, such that* $d(Jx, Jy) \leq L\, d(x, y)$ *, for all x, y* $\in$ *X and a Lipschitz constant L‹1,then J has one, and only one, fixed point* $x^* = J(x^*) = Jx^*$*, such that* $\lim_{n\to\infty} J^n x = x^*$ *for any starting point* $x \in X$*, as well as* $d(J^n x, x^*) \leq L^n\, d(x, x^*)$*, for all* $n \geq 0$ *and* $x \in X$ *;* $d(J^n x, x^*) \leq \frac{1}{1-L} d(J^n x, J^{n+1} x)$*, for all* $n \geq 0$ *and* $x \in X$ *; and* $d(x, x^*) \leq \frac{1}{1-L} d(x, Jx)$*, for all* $x \in X$.

We note that $d(Jx, x^*) = d(Jx, Jx^*) \leq L\, d(x, x^*)$. Thus
$d(J^2 x, x^*) = d(J^2 x, Jx^*) = d(J^2 x, J^2 x^*) = d(J\,Jx, J\,Jx^*) \leq L\, d(Jx, Jx^*) \leq L^2\, d(x, x^*)$.
Thus by induction on $n \in \mathbb{N}$ we get $d(J^n x, x^*) \leq L^n\, d(x, x^*)$.

Alternative Contraction Principle 1.2 ([2]). *If* (X, d) *is a complete metric space and* J: $X \to X$ *a strictly contractive mapping (that is :* $d(Jx, Jy) \leq L\, d(x, y)$ *, for all x, y* $\in$ *X and a Lipschitz constant L* (‹*1*) *), then either* $d(J^n x, J^{n+1} x) = \infty$*, for all* $n \geq 0$ *, or there exists*

$n_0 \in N = \{1, 2, ...\}$ *such that* $d\left(J^n x, J^{n+1}x\right) \langle \infty$, *for all* $n \geq n_0$; *the sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J; y^* *is the unique fixed point of J in the set* $Y = \left\{ y \in X, d\left(J^{n_0}x, y\right) \langle \infty \right\}$; *and* $d\left(y, y^*\right) \leq \dfrac{1}{1-L} d\left(y, Jy\right)$, *for all* $y \in Y$.

2. STABILITY OF THE JENSEN AND JENSEN TYPE EQUATION

Throughout this paper, let E be a linear space and F a Banach space. Besides denote $\delta^* = \delta + \| f(0) \|$, and $\delta^{**} = \delta^* + \| f(0) \|$ for some fixed $\delta \geq 0$ and $f: E \to F$.

We prove the following Theorem 2.1 on Jensen mappings via the alternative contraction principle.

Theorem 2.1. *If* $f: E \to F$ *satisfies the approximately Jensen inequality*

$$\left\| 2f\left(\frac{x_1 + x_2}{2}\right) - f\left(x_1\right) - f\left(x_2\right) \right\| \leq \delta \tag{2.1}$$

for some fixed $\delta \geq 0$ *and all* $x_1, x_2 \in E$, *then there exists a unique Jensen mapping* $j: E \to F$ *satisfying the inequality*

$$\| f(x) - j(x) \| \leq \delta^* \left(= \frac{L}{1 - L} \delta^* \quad for \quad L = \frac{1}{2} \langle 1 \right), \tag{2.2}$$

for all $x \in E$.

Proof. Let $X = \{ g : E \to F \}$, and introduce *the metric*

$$d(g,h) = \inf\left\{ \varepsilon \in \mathbb{R}^+ = (0,\infty), \ \|g(x) - h(x)\| \leq \delta^* \varepsilon, \ \forall x \in E \right\} \quad (\langle \varepsilon) \tag{2.3}$$

for all g, h $\in$ X.

It is clear that (X, d) is *complete.*

Consider the linear mapping $J: X \to X$, such that $Jg(x) = 2^{-1} g(2x)$; thus,

$$J^n g(x) = 2^{-n} g\left(2^n x\right).$$

Replacing $2x$ on x in the above inequality $\| g(x) - h(x) \| \leq \delta^* \varepsilon$

of the metric (2.3), and then dividing by 2, one gets

$$\|Jg(x)-Jh(x)\| = 2^{-1}\|g(2x)-h(2x)\| \le 2^{-1}\delta^{*}\varepsilon \text{, or } d(Jg, Jh) \le L\varepsilon \text{ (for } L=1/2\text{)},$$

yielding $d(Jg, Jh) \le L\, d(g,h)$,

for all $g, h \in X$. Thus $J{:}X \to X$ is *strictly contractive* with Lipschitz constant L=1/2.

Setting $x_1 = 2x$, $x_2 = 0$ in (2.1) and employing the triangle inequality, we obtain

$$\|\, 2f(x) - f(2x) - f(0)\,\| \le \delta \text{, or}$$

$$\|f(x) - Jf(x)\| = \|f(x) - 2^{-1}f(2x)\| \le (\delta + \|f(0)\|)/2 = \delta^{*}/2 = \delta^{*}L\,.$$

Thus $d(f, Jf) \le L$.

Placing $x_1 = x$, $x_2 = -x$ in (2.1) we get

$$\|\, 2f(0) - f(x) - f(-x)\,\| \le \delta \text{, or}$$

$$\|f(-x) + f(x)\| \le \delta + 2\|f(0)\| = \delta^{**} = \theta$$

for all $x \in E$.

Employing *the alternative contraction principle 1.2*, one establishes *the existence of* a Jensen mapping $j: X \to X$ such that $j(x) = 2^{-1} j(2x)$; that is, $j: X \to X$ is *a fixed point of* $J: X \to X$,such that $Jj(x) = j(x) = 2^{-1} j(2x)$ in $Y = \{ g \in X, d(f,g) \langle \infty \}$.

Thus j: $X \to X$ is *the unique mapping* satisfying $j(x) = 2^{-1} j(2x)$ and the inequality

$$\|\, f(x) - j(x)\| \le \delta^{*}\varepsilon \;\; (= \delta^{*} \;\; if \;\; \varepsilon = 1),$$

for all $x \in E$.

Besides

$$0 \le d(J^n f, j) \le L^n d(f, j) = 2^{-n}\varepsilon \to 0 \text{, as } n \to \infty \text{, or } \lim_{n\to\infty} d(J^n f, j) = 0\,.$$

This implies

(2.4) $$\lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) = j(x) \;,$$

for all $x \in E$.

We note that $Jf(x) = 2^{-1} f(2x)$ yields

$$J^2 f(x) = J(Jf(x)) = J(2^{-1} f(2x)) = 2^{-1}2^{-1} f(2(2x)) = 2^{-2} f(2^2 x).$$

Thus by induction on $n \in \mathbb{N}$, one gets $J^n f(x) = 2^{-n} f(2^n x), \quad \forall n \in \mathbb{N}$.

Therefore from triangle inequality, we find

$$\begin{aligned}
&\left\| f(x) - J^n f(x) \right\| \left(= \left\| f(x) - 2^{-n} f(2^n x) \right\| \right) \\
&\leq \left\| f(x) - Jf(x) \right\| \\
&+ \left\| Jf(x) - J^2 f(x) \right\| + \ldots + \left\| J^{n-1} f(x) - J^n f(x) \right\| \\
&\leq \frac{\delta^*}{2} \left(1 + 2^{-1} + \ldots + 2^{-(n-1)} \right) \\
&= \delta^* \left(1 - 2^{-n} \right) \to \delta^*, \; as \;\; n \to \infty .
\end{aligned}$$

Therefore

$$\left\| f(x) - j(x) \right\| \leq \left\| f(x) - J^n f(x) \right\| + \left\| J^n f(x) - j(x) \right\| \to \delta^* + 0 = \delta^*, \;\; n \to \infty ,$$

satisfying the inequality (2.2). Also

$$d(f, j) \leq \frac{1}{1-L} d(f, df) \leq \frac{L}{1-L} = 1 .$$

Replacing $(2^n x_1, 2^n x_2)$ on (x_1, x_2) in (2.1), and multiplying by 2^{-n}, we get from (2.4) by letting $n \to \infty$, that

$$\left\| \, 2 \cdot 2^{-n} f\left(2^n \left(\frac{x_1 + x_2}{2} \right) \right) - 2^{-n} f\left(2^n x_1\right) - 2^{-n} f\left(2^n x_2\right) \right\| \leq \; 2^{-n} \delta \to 0, as \;\; n \to \infty ,$$

or that $j : X \to X$ is a Jensen mapping, satisfying *the Jensen equation*

(2.5) $$2j\left(\frac{x_1 + x_2}{2} \right) = j(x_1) + j(x_2),$$

for all $x_1, x_2 \in E$, completing the proof of Theorem 2.1.

Corollary 2.1. *If $f : E \to F$ satisfies the approximately Jensen inequality (2.1), then there exists a unique additive mapping $j : E \to F$ satisfying the inequality (2.2).*

Proof. Following the proof of Theorem 2.1 we get that there exists a unique Jensen mapping $j : E \to F$ satisfying the Jensen equation (2.5).

However, if we replace $(2^{n+1} x_1, 2^{n+1} x_2)$ on (x_1, x_2) in (2.1), and multiply by $2^{-(n+1)}$, we get from (2.4) by letting $n \to \infty$, that

$$\left\| 2\cdot 2^{-(n+1)} f\left(2^{n+1}\left(\frac{x_1+x_2}{2}\right)\right) - 2^{-(n+1)} f\left(2^{n+1}x_1\right) - 2^{-(n+1)} f\left(2^{n+1}x_2\right) \right\|$$

$$= \left\| 2^{-n} f\left(2^n (x_1+x_2)\right) - 2^{-(n+1)} f\left(2^{n+1}x_1\right) - 2^{-(n+1)} f\left(2^{n+1}x_2\right) \right\|$$

$$\leq 2^{-(n+1)}\delta \to 0, as\ n \to \infty, \text{ or}$$

$$0 \leq \left\| j(x_1+x_2) - j(x_1) - j(x_2) \right\| \leq 0, \text{ or}$$

$$j(x_1+x_2) = j(x_1) + j(x_2) \ ;$$

that is, *a unique additive mapping* $j : E \to F$, satisfying (2.2), and thus the proof of this Corollary 2.1 is complete.

Claim that a Jensen mapping $j : E \to F$ *with* $j(0) = 0$ *is additive.*

In fact, we take $x_1 = 2x$, $x_2 = 0$ in (2.5). Then

$$2j(x) = j(2x) + j(0) = j(2x), \text{ or}$$

$$j(2x) = 2j(x), \tag{2.6}$$

for all $x \in X$. Thus replacing $(2x_1, 2x_2)$ on (x_1, x_2) in (2.5) we find from (2.6) that

$$j(x_1+x_2) = j(x_1) + j(x_2),$$

for all $x_1, x_2 \in X$. Therefore $j : E \to F$ is additive.

Similarly we establish the following Theorem 2.2.

Theorem 2.2. *If* $f : E \to F$ *satisfies the approximately Jensen type inequality*

$$\left\| 2f\left(\frac{x_1-x_2}{2}\right) - f(x_1) + f(x_2) \right\| \leq \delta \tag{2.7}$$

for some fixed $\delta \geq 0$ *and all* $x_1, x_2 \in E$,

then there exists a unique Jensen type mapping

$j : E \to F$ *satisfying the inequality (2.2).*

Corollary 2.2. *If* $f : E \to F$ *satisfies the approximately Jensen type inequality (2.7), then there exists a unique additive mapping* $j : E \to F$ *satisfying the inequality (2.2).*

3. STABILITY OF THE ADDITIVE EQUATION (1.1) OF THE FIRST FORM

Theorem 3.1. *If $f: E \to F$ satisfies the approximately additive inequality of the first form*

(3.1) $$\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) \| \leq \delta$$

for some fixed $\delta \geq 0$ and all x_1, $x_2 \in E$, then there exists a unique additive mapping $j: E \to F$ of the first form satisfying the inequality

(3.2) $$\| f(x) - j(x) \| \leq \delta^* \left(= \frac{L}{1 - L} \delta^* \ \ for\ \ L = \frac{1}{2} \right),$$

for all $x \in E$.

Proof. In fact, we set $x_1 = x_2 = x$ in (3.1) and $Jf(x) = 2^{-1} f(2x)$, one gets that

(3.3) $$\| f(x) - Jf(x) \| = \| f(x) - 2^{-1} f(2x) \| \leq \frac{\delta^*}{2} = \delta^* L,$$

for all $x \in E$. Replacing $(2^n x_1, 2^n x_2)$ on (x_1, x_2) in (3.1), dividing by 2^n, and considering the formula $j(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$, we get the following additive equation of the first form

(3.4) $$j(x_1 + x_2) + j(x_1 - x_2) = 2j(x_1),$$

for all x_1, $x_2 \in E$. Thus $j: E \to F$ is additive of the first form.The rest of the proof is omitted as similar to the proof of the afore-mentioned Theorem 2.1.

Corollary 3.1. *If $f: E \to F$ satisfies the condition $f(0) = 0$ and the approximately additive inequality of the first form (3.1), then there exists a unique additive mapping $j: E \to F$ satisfying the condition $j(0) = 0$ and the inequality.*

(3.5) $$\| f(x) - j(x) \| \leq \delta$$

for all $x \in E$ and some fixed $\delta \geq 0$.

Proof. Following the proof of Theorem 3.1. we get that there exists a unique additive mapping $j: E \to F$ of the first form satisfying the equation (3.4) and the inequality (3.5). Besides $j(0) = 0$ because $f(0) = 0$.

Claim that an additive mapping $j: E \to F$ of the first form with $j(0) = 0$ is additive.

In fact, setting $x_1 = 0$, $x_2 = x$ in the equation (3.4) and assuming $j(0) = 0$, we get that $j: E \to F$ is odd; that is, (3.6) $$j(-x) = -j(x),$$

for all $x \in X$. Besides replacing (x_2, x_1) on (x_1, x_2) in (3.4), one obtains from (3.6) that

(3.7) $$j(x_1 + x_2) - j(x_1 - x_2) = 2j(x_2)$$

for all $x_1, x_2 \in E$. Thus adding (3.4) and (3.7), we find

(3.8) $$j(x_1 + x_2) = j(x_1) + j(x_2)$$

for all $x_1, x_2 \in E$, completing the proof of the additivity of $j : E \to F$.

4. STABILITY OF THE ADDITIVE EQUATION (1.4) OF THE SECOND FORM

Theorem 4.1. *If f: $E \to F$ satisfies the approximately additive inequality of the second form*

(4.1) $$\| f(x_1 + x_2) - f(x_1 - x_2) - 2f(x_2) \| \le \delta$$

for some fixed $\delta \ge 0$ and all $x_1, x_2 \in E$, then there exists a unique additive mapping j: $E \to F$ of the second form satisfying the inequality (3.2).

Proof. In fact, we set $x_1 = x_2 = 0$ in (4.1) yielding

(4.2) $$\| f(0) \| \le \frac{\delta}{2}$$

for some fixed $\delta \ge 0$.

Besides placing (x, x) on (x_1, x_2) in (4.1) we find from (4.2), the triangle inequality and $Jf(x) = 2^{-1} f(2x)$ that

$$\| f(2x) - f(0) - 2f(x) \| \le \delta ,$$

or (3.3) holds. Replacing $(2^n x_1, 2^n x_2)$ on (x_1, x_2) in (4.1), dividing by 2^n,

and considering $j(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, we get the additive equation of the second form:

(4.3) $$j(x_1 + x_2) - j(x_1 - x_2) = 2j(x_2) ,$$

for all $x_1, x_2 \in E$. Thus j: $E \to F$ is additive of the second form. We note that if we set $x_1 = x_2 = 0$ in (4.3), one finds that $j(0) = 0$.The rest of the proof is omitted as similar to the proof of the above-mentioned Theorem 2.1.

Corollary 4.1. *If* $f : E \to F$ *satisfies the approximately additive inequality of the second form (4.1), then there exists a unique additive mapping* $j: E \to F$ *satisfying the inequality (3.2).*

Proof. Following the proof of the Theorem 4.1 we get that there exists a unique additive mapping $j: E \to F$ of the second form satisfying the equation (4.3) and the inequality (3.2).

Claim that an additive mapping $j : E \to F$ *of the second form is additive.*

In fact, setting $x_1 = 0$, $x_2 = x$ in the equation (4.3), we get (3.6).

Besides replacing (x_2, x_1) on (x_1, x_2) in (4.3), one obtains (3.4) from (3.6). Thus adding (3.4) and (4.3), we find (3.8), completing the proof of the additivity of $j: E \to F$.

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