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On the Ulam stability of Jensen and Jensen type mappings on restricted domains

John Michael Rassias * and Matina John Rassias

Pedagogical Department, Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4 Agamemnonos Str., Aghia Paraskevi, Athens 15342, Greece

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Abstract

In 1941 Hyers solved the well-known Ulam stability problem for linear mappings. In 1951 Bourgin was the second author to treat this problem for additive mappings. In 1982–1998 Rassias established the Hyers–Ulam stability of linear and nonlinear mappings. In 1983 Skof was the first author to solve the same problem on a restricted domain. In 1998 Jung investigated the Hyers–Ulam stability of more general mappings on restricted domains. In this paper we introduce additive mappings of two forms: of "Jensen" and "Jensen type," and achieve the Ulam stability of these mappings on restricted domains. Finally, we apply our results to the asymptotic behavior of the functional equations of these types.

Keywords: Ulam stability problem; Hyers–Ulam stability; Ulam stability; Approximately odd mapping; Additive of the first form; Additive of the second form; Jensen equation; Jensen type equation; Restricted domain

1. Introduction

In 1940 and in 1968 Ulam [24] proposed the *general Ulam stability problem*:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

* Corresponding author.
E-mail address: jrassias@primedu.uoa.gr (J.M. Rassias).

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In 1941 Hyers [13] solved the aforementioned problem for linear mappings. In 1951 Bourgin [3] was the second author to treat this problem for additive mappings. In 1978, according to Gruber [12], this kind of stability problems is of particular interest in probability theory and in functional equations. In 1978 Rassias [22] employed Hyers' ideas to new linear mappings. In 1980 and in 1987, Fenyö [7,8] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Gajda and Ger [10] showed that one can get analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors: Aczél [1], Borelli and Forti [2,9], Cholewa [4], Czerwik [5], Drljevic [6], and Kannappan [15]. In 1982–1998 Rassias [16–21] established the Hyers–Ulam stability of linear and nonlinear mappings. In 1999 Gavruta [11] answered a question of Rassias [18] concerning the stability of the Cauchy equation. In 1983 Skof [23] was the first author to solve the Ulam problem on a restricted domain. In 1998 Jung [14] investigated the Hyers–Ulam stability for more general mappings on restricted domains. In this paper we introduce additive mappings of two new forms: of "Jensen" and "Jensen type," and achieve the Ulam stability of these mappings on restricted domains. Finally, we apply our results to the asymptotic behavior of the functional equations of these types.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well let X and Y be real linear spaces for functional equations.

Definition 1. A mapping $A : X \to Y$ is called *additive of the first form* if A satisfies the additive functional equation

$$A(x_1 + x_2) + A(x_1 - x_2) = 2A(x_1) \tag{1}$$

for all $x_1, x_2 \in X$. We note that (1) is equivalent to *the Jensen equation*

$$A\left(\frac{x+y}{2}\right) = \frac{1}{2}\big[A(x) + A(y)\big] \tag{1$'$}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$.

Definition 2. A mapping $A : X \to Y$ is called *additive of the second form* if A satisfies the additive functional equation

$$A(x_1 + x_2) - A(x_1 - x_2) = 2A(x_2) \tag{2}$$

for all $x_1, x_2 \in X$. We note that (2) is equivalent to *the Jensen type equation*

$$A\left(\frac{x-y}{2}\right) = \frac{1}{2}\big[A(x) - A(y)\big] \tag{2$'$}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$.

Definition 3. A mapping $f : X \to Y$ is called *approximately odd* if f satisfies the functional inequality

$$\big\| f(x) + f(-x) \big\| \leqslant \theta \tag{3}$$

for some fixed $\theta \geqslant 0$ and for all $x \in X$.

In this section we state the following Theorem 1 which was proved by Rassias [19] in 1994.

Theorem 1. *If a mapping $f : X \to Y$ satisfies the inequalities*

$$\left\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1)\right\| \leqslant \delta, \tag{4}$$

$$\left\| f(0)\right\| \leqslant \delta_0 \tag{5}$$

for some fixed $\delta, \delta_0 \geqslant 0$ and for all $x_1, x_2 \in X$, then there exists a unique additive mapping $A : X \to Y$ of the first form which satisfies the inequality

$$\left\| f(x) - A(x)\right\| \leqslant \delta + \delta_0 \tag{6}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in R$.

The last assertion holds according to Rassias' work [16] in 1982.

2. Stability of Eq. (1) on a restricted domain

Theorem 2. *Let $d > 0$ and $\delta, \delta_0 \geqslant 0$ be fixed. If a mapping $f : X \to Y$ satisfies inequality* (4) *for all $x_1, x_2 \in X$, with $\|x_1\| + \|x_2\| \geqslant d$, and* (5)*, then there exists a unique additive mapping $A : X \to Y$ of the first form such that*

$$\left\| f(x) - A(x)\right\| \leqslant \frac{5}{2}\delta + \delta_0 \tag{7}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in R$.

Proof. Assume $\|x_1\| + \|x_2\| < d$. If $x_1 = x_2 = 0$, then we choose $t \in X$ with $\|t\| = d$. Otherwise, let us choose

$$t = \left(1 + \frac{d}{\|x_1\|}\right)x_1 \quad \text{if } \|x_1\| \geqslant \|x_2\|, \qquad t = \left(1 + \frac{d}{\|x_2\|}\right)x_2 \quad \text{if } \|x_1\| \leqslant \|x_2\|.$$

We note that $\|t\| = \|x_1\| + d > d$ if $\|x_1\| \geqslant \|x_2\|$, $\|t\| = \|x_2\| + d > d$ if $\|x_1\| \leqslant \|x_2\|$. Clearly, we see that

$$\begin{aligned} &\|x_1 - t\| + \|x_2 + t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d, \qquad \|x_1 - x_2\| + \|2t\| \geqslant d, \\ &\|x_1 + t\| + \|-x_2 + t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d, \qquad \|x_1\| + \|t\| \geqslant d. \end{aligned} \tag{8}$$

Inequalities (8) come from the corresponding substitutions attached between the right-hand sided parentheses of the following functional identity.

Therefore from (4), (8), the triangle inequality, and the *functional identity*

$$\begin{aligned} &2\big[f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1)\big] \\ &\quad = \big[f(x_1 + x_2) + f(x_1 - x_2 - 2t) - 2f(x_1 - t)\big] \end{aligned}$$

$$
\begin{aligned}
&\quad \text{(with } x_1 - t \text{ on } x_1 \text{ and } x_2 + t \text{ on } x_2\text{)} \\
&- \big[f(x_1 - x_2 - 2t) + f(x_1 - x_2 + 2t) - 2f(x_1 - x_2) \big] \\
&\quad \text{(with } x_1 - x_2 \text{ on } x_1 \text{ and } 2t \text{ on } x_2\text{)} \\
&+ \big[f(x_1 - x_2 + 2t) + f(x_1 + x_2) - 2f(x_1 + t) \big] \\
&\quad \text{(with } x_1 + t \text{ on } x_1 \text{ and } -x_2 + t \text{ on } x_2\text{)} \\
&+ 2\big[f(x_1 + t) + f(x_1 - t) - 2f(x_1) \big] \quad \text{(with } x_1 \text{ on } x_1 \text{ and } t \text{ on } x_2\text{)},
\end{aligned}
$$

we get

$$2\big\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) \big\| \leqslant \delta + \delta + \delta + 2\delta = 5\delta$$

or

$$\big\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) \big\| \leqslant \frac{5}{2}\delta. \tag{9}$$

Applying now Theorem 1 and inequality (9), one gets that there exists a unique additive mapping $A : X \to Y$ of the first form that satisfies the additive equation (1) and inequality (7), such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$. Our last assertion is trivial according to Theorem 1.

We note that if we define $S_2 = \{(x_1, x_2) \in X^2\colon \ \|x_i\| < d,\ i = 1, 2\}$ for some $d > 0$, then $\{(x_1, x_2) \in X^2\colon \ \|x_1\| + \|x_2\| \geqslant 2d\} \subset X^2 \backslash S_2$. □

Corollary 1. *If we assume that a mapping $f : X \to Y$ satisfies inequalities* (4)–(5) *for some fixed $\delta, \delta_0 \geqslant 0$ and for all $(x_1, x_2) \in X^2 \backslash S_2$, then there exists a unique additive mapping $A : X \to Y$ of the first form, satisfying* (7) *for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and all $t \in R$.*

Corollary 2. *A mapping $f : X \to Y$ is additive of the first form if and only if the asymptotic condition*

$$\big\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) \big\| \to 0 \quad \textit{as } \|x_1\| + \|x_2\| \to \infty \tag{10}$$

holds.

Proof. Following the corresponding techniques of the proof of Jung [14], one gets from Theorem 2 and asymptotic condition (10) that f is additive of the first form. The reverse assertion is obvious. □

However, in 1983 Skof [23] proved an asymptotic property for *the additive mappings* $A : X \to Y$, such that

$$A(x_1 + x_2) = A(x_1) + A(x_2) \tag{11}$$

holds for all $x_1, x_2 \in X$.

3. Stability of Eq. (2)

Theorem 3. *If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| f(x_1 + x_2) - f(x_1 - x_2) - 2f(x_2) \right\| \leqslant \delta \tag{12}$$

for some $\delta \geqslant 0$ and for all $x_1, x_2 \in X$, then there exists a unique additive mapping $A : X \to Y$ of the second form which satisfies the inequality

$$\left\| f(x) - A(x) \right\| \leqslant \frac{3}{2}\delta \tag{13}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in R$.

Proof. Replacing $x_1 = x_2 = 0$ in (12), we find

$$\left\| f(0) \right\| \leqslant \frac{\delta}{2}. \tag{14}$$

Thus, substituting $x_1 = x_2 = x$ in (12), one gets

$$\left\| f(2x) - f(0) - 2f(x) \right\| \leqslant \delta \quad \text{or} \quad \left\| f(2x) - 2f(x) \right\| \leqslant \delta + \left\| f(0) \right\| \leqslant \frac{3}{2}\delta, \quad \text{or}$$

$$\left\| f(x) - 2^{-1} f(2x) \right\| \leqslant \frac{3}{2}\delta\left(1 - 2^{-1}\right) \tag{15}$$

for all $x \in X$. Therefore from (15), with $2^i x$ on place of x $(i = 1, 2, \ldots, n-1)$, we obtain

$$\begin{aligned} \left\| f(x) - 2^{-n} f(2^n x) \right\| &\leqslant \left\| f(x) - 2^{-1} f(2x) \right\| + \left\| 2^{-1} f(2x) - 2^{-2} f(2^2 x) \right\| + \cdots \\ &\quad + \left\| 2^{-(n-1)} f(2^{n-1} x) - 2^{-n} f(2^n x) \right\| \\ &\leqslant \frac{3}{2}\delta\left(1 + 2^{-1} + \cdots + 2^{-(n-1)}\right)\left(1 - 2^{-1}\right) \end{aligned}$$

or

$$\left\| f(x) - 2^{-n} f(2^n x) \right\| \leqslant \frac{3}{2}\delta\left(1 - 2^{-n}\right) \tag{16}$$

for any $n \in N$ and all $x \in X$.

We claim that

$$A(x) = 2^{-n} A(2^n x) \tag{17}$$

holds for any $n \in N$ and all $x \in X$. In fact, replacing $x_1 = x_2 = 0$ in (2) one finds $A(0) = 0$. Thus substituting $x_1 = x_2 = x$ in (2) we get $A(2x) = 2A(x)$ for all $x \in X$. Therefore by induction on n one gets that

$$A(2^{n+1} x) = A(2 \cdot 2^n x) = 2A(2^n x) = 2 \cdot 2^n A(x) = 2^{n+1} A(x)$$

for all $x \in X$, completing the proof of (17).

By (16), for $n \geqslant m > 0$ and $h = 2^m x$, we have

$$\begin{aligned}\left\|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\right\| &= 2^{-m}\left\|2^{-(n-m)} f(2^{n-m} h) - f(h)\right\| \\ &\leqslant 2^{-m}\frac{3}{2}\delta\left(1 - 2^{-(n-m)}\right) = \frac{3}{2}\delta\left(2^{-m} - 2^{-n}\right) < \frac{3}{2}\delta 2^{-m} \to 0 \quad \text{as } m \to \infty. \end{aligned} \tag{18}$$

From (18) and the completeness of Y, we get that the Cauchy sequence $\{2^{-n} f(2^n x)\}$ converges. Therefore we may apply a direct method to the definition of A, such that the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{19}$$

holds for all $x \in X$ [16–19]. From formula (19) and inequality (12), it follows that

$$\begin{aligned}&\left\|A(x_1 + x_2) - A(x_1 - x_2) - 2A(x_2)\right\| \\ &\quad = \lim_{n\to\infty} 2^{-n}\left\|f(2^n x_1 + 2^n x_2) - f(2^n x_1 - 2^n x_2) - 2f(2^n x_2)\right\| \leqslant \lim_{n\to\infty} 2^{-n}\delta = 0,\end{aligned}$$

or Eq. (2) holds for all $x_1, x_2 \in X$. Thus $A : X \to Y$ is an additive mapping of the second form. According to inequality (16) and formula (19), one gets that inequality (13) holds.

Assume now that there is another additive mapping $A' : X \to Y$ of the second form which satisfies Eq. (2), formula (17), and inequality (13). Therefore

$$\begin{aligned}\left\|A(x) - A'(x)\right\| &= 2^{-n}\left\|A(2^n x) - A'(2^n x)\right\| \\ &\leqslant 2^{-n}\left[\left\|A(2^n x) - f(2^n x)\right\| + \left\|f(2^n x) - A'(2^n x)\right\|\right] \\ &\leqslant 2^{-n}\left(\frac{3}{2}\delta + \frac{3}{2}\delta\right) = 3\delta 2^{-n} \to 0 \quad \text{as } n \to \infty\end{aligned}$$

or

$$A(x) = A'(x) \tag{20}$$

for all $x \in X$, completing the proof of the first part of our Theorem 3.

The proof of the last assertion in our Theorem 3 is obvious according to Rassias' work [16]. □

4. Stability of Eq. (2) on a restricted domain

We note that from (3) and $\|f(-2x) + f(2x)\| \leqslant \theta$ (from (3) with $2x$ on x) and (15) as well as $\|f(-2x) - 2f(-x)\| \leqslant (3/2)\delta$ (from (15) with $2x$ on x), and the triangle inequality one gets

$$\begin{aligned}2\left\|f(-x) + f(x)\right\| &\leqslant \left\|-\left[f(-2x) - 2f(-x)\right]\right\| \\ &\quad + \left\|-\left[f(2x) - 2f(x)\right]\right\| + \left\|f(-2x) + f(2x)\right\| \\ &\leqslant \frac{3}{2}\delta + \frac{3}{2}\delta + \theta = 3\delta + \theta\end{aligned}$$

or

$$\left\| f(-x) + f(x) \right\| \leqslant \frac{3}{2}\delta + \frac{\theta}{2} \overset{(3)}{=} \theta.$$

Therefore $\theta = 3\delta$ and (3) takes the independent of θ equivalent form

$$\left\| f(-x) + f(x) \right\| \leqslant 3\delta. \tag{3'}$$

Theorem 4. *Let $d > 0$ and $\delta \geqslant 0$ be fixed. If an approximately odd mapping $f : X \to Y$ satisfies inequality* (12) *for all $x_1, x_2 \in X$ with $\|x_1\| + \|x_2\| \geqslant d$ and inequality* (3′) *for all $x \in X$ with $\|x\| \geqslant d$, then there exists a unique additive mapping $A : X \to Y$ of the second form such that*

$$\left\| f(x) - A(x) \right\| \leqslant \frac{33}{2}\delta \tag{21}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in R$.

Proof. Assume $\|x_1\| + \|x_2\| < d$. If $x_1 = x_2 = 0$, then we choose $t \in X$ with $\|t\| = d$. Otherwise, let us choose

$$t = \left(1 + \frac{d}{\|x_1\|}\right)x_1 \quad \text{if } \|x_1\| \geqslant \|x_2\|, \qquad t = \left(1 + \frac{d}{\|x_2\|}\right)x_2 \quad \text{if } \|x_1\| \leqslant \|x_2\|.$$

We note that $\|t\| = \|x_1\| + d > d$ if $\|x_1\| \geqslant \|x_2\|$, $\|t\| = \|x_2\| + d > d$ if $\|x_1\| \leqslant \|x_2\|$. Clearly, we see that

$$\begin{aligned}
&\|x_1 - t\| + \|x_2 + t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d,\\
&\|x_1 - t\| + \|x_2 - t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d,\\
&\|x_1 - 2t\| + \|x_2\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d, \qquad \|t\| + \|x_2\| \geqslant d,
\end{aligned} \tag{22}$$

and $\|t - x_2\| \geqslant \|t\| - \|x_2\| = (\|x_2\| + d) - \|x_2\| = d$, because $\|t\| = \|x_2\| + d$.

Therefore from (3′), (12), (22), and the functional identity

$$\begin{aligned}
&f(x_1 + x_2) - f(x_1 - x_2) - 2f(x_2)\\
&\quad = \big[f(x_1 + x_2) - f(x_1 - x_2 - 2t) - 2f(x_2 + t)\big]\\
&\qquad\qquad \text{(with } x_1 - t \text{ on } x_1 \text{ and } x_2 + t \text{ on } x_2\text{)}\\
&\qquad + \big[f(x_1 + x_2 - 2t) - f(x_1 - x_2) - 2f(x_2 - t)\big]\\
&\qquad\qquad \text{(with } x_1 - t \text{ on } x_1 \text{ and } x_2 - t \text{ on } x_2\text{)}\\
&\qquad - \big[f(x_1 + x_2 - 2t) - f(x_1 - x_2 - 2t) - 2f(x_2)\big]\\
&\qquad\qquad \text{(with } x_1 - 2t \text{ on } x_1 \text{ and } x_2 \text{ on } x_2\text{)}\\
&\qquad + 2\big[f(t + x_2) - f(t - x_2) - 2f(x_2)\big]\\
&\qquad\qquad \text{(with } t \text{ on } x_1 \text{ and } x_2 \text{ on } x_2\text{)}\\
&\qquad + 2\big[f(t - x_2) + f\big(-(t - x_2)\big)\big] \quad \text{(with } t - x_2 \text{ on } x\text{)},
\end{aligned}$$

we get

$$\left\| f(x_1+x_2) - f(x_1-x_2) - 2f(x_2) \right\| \leqslant \delta + \delta + \delta + 2\delta + 6\delta = 5\delta + 6\delta = 11\delta. \tag{23}$$

Applying Theorem 3 and inequality (23), we prove that there exists a unique additive mapping $A : X \to Y$ of the second form that satisfies Eq. (2) and inequality (21), completing the proof of Theorem 4. □

We note that if we define $S_1 = \{x \in X\colon \|x\| < d\}$ and $S_2 = \{(x_1, x_2) \in X^2\colon \|x_i\| < d, i = 1, 2\}$ for some fixed $d > 0$, then $\{x \in X\colon \|x\| \geqslant 2d\} \subset X \backslash S_1$ and $\{(x_1, x_2) \in X^2\colon \|x_1\| + \|x_2\| \geqslant 2d\} \subset X^2 \backslash S_2$.

Corollary 3. *If we assume that a mapping $f : X \to Y$ satisfies inequality* (12) *for some fixed $\delta \geqslant 0$ and* (3′) *for all $x \in X \backslash S_1$ and for all $(x_1, x_2) \in X^2 \backslash S_2$, then there exists a unique additive mapping $A : X \to Y$ of the second form, satisfying* (21) *for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in R$.*

Corollary 4. *A mapping $f : X \to Y$ is additive of the second form if and only if the asymptotic conditions*

$$\left\| f(-x) + f(x) \right\| \to 0 \quad \textit{and} \quad \left\| f(x_1+x_2) - f(x_1-x_2) - 2f(x_2) \right\| \to 0, \tag{24}$$

as $\|x\| \to \infty$ and $\|x_1\| + \|x_2\| \to \infty$ hold, respectively.

Proof. Following the corresponding techniques of the proof of Jung [14], one gets from Theorem 4 and asymptotic conditions (24) that f is additive of the second form. The reverse assertion is clear. □

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