Southeast Asian Bulletin of Mathematics (2002) 26: 483–502

Southeast Asian
Bulletin of
Mathematics

On the Hyers-Ulam Stability Problem for Quadratic Multi-dimensional Mappings on the Gaussian Plane

John Michael Rassias

Pedagogical Department, E. E., National and Capodistrian University of Athens, Section of Mathematics and Informatics, 4, Agamemnonos Str., Aghia Paraskevi, Athens 15342, Greece
E-mail: jrassias@primedu.uoa.gr

AMS Subject Classification (1991): 39B

Abstract. In this paper we solve the Hyers-Ulam stability problem for quadratic multi-dimensional mappings on the Gaussian plane.

Keywords: Ulam Problem, Hyers-Ulam Stability, Quadratic Weighted Means, Gaussian Plane, Quadratic Mapping, Mean Equation, Fundamental Equation

1. Introduction

In 1968 S.M. Ulam [17] proposed *the general stability problem*: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?". According to P.M. Gruber [3] this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982–1999 we ([5]–[15]) solved the above Ulam problem for different mappings. In this paper we first introduce *new quadratic weighted means and fundamental functional equations* and then solve the Hyers-Ulam stability problem ([4], [17]–[18]) for quadratic mappings $Q : Z \to W$ *on the Gaussian plane* C, satisfying a mean equation and the functional equation

$$Q\left(\sum_{i=1}^{n} a_i z_i\right) + \sum_{1 \le i < j \le n} Q(a_j \bar{z}_i - a_i \bar{z}_j) = m \sum_{i=1}^{n} Q(z_i),$$

for every complex $z_i \in Z$ $(i = 1, 2, \ldots, n)$, and fixed complex a_i $(i = 1, 2, \ldots, n)$:

$$0 < m = \sum_{i=1}^{n} |a_i|^2 \neq \left[1 + \binom{n}{2}\right] \Big/ n,$$

where Z and W are complex linear spaces (on a field of characteristic $\neq 2$). We note that in 1996–1998 we ([13], [14]) established the corresponding case $n = 2$ on the real plane R^2.

To the best of our knowledge the afore-mentioned functional equations are established for the first time for $n > 2$ and are the most general quadratic functional equations of this kind till now.

2. Fundamental Functional Equation of First Type

Let Z and W be complex linear spaces (on a field of characteristic $\neq 2$). Then we consider a non-linear mapping $Q : Z \to W$ satisfying *the fundamental functional equation of first type*

$$Q(a_1 z) + \sum_{j=2}^{n} Q(a_j \bar{z}) = \sum_{i=1}^{n} Q(\bar{a}_i z), \qquad (*)$$

($\bar{z}$ = conjugate of z, and $\bar{a}_i$ = conjugate of a_i), with

$$1 < m = \sum_{i=1}^{n} |a_i|^2 \neq \left[1 + \binom{n}{2}\right] \Big/ n, \quad n = 2, 3, 4, \ldots,$$

and every $z \in Z$, as well as any fixed $a = (a_1, a_2, \ldots, a_n) \in C^n$, $a \neq 0$.

We note that if z_i, a_i $(i = 1, 2) \in R$, then

$$m = \sum_{i=1}^{2} a_i^2 > 1,$$

and equation $(*)$ is an identity in Z.

Thus $(*)$ is not required.

Definition 2.1. Let Z and W be linear spaces (*on a field of characteristic* $\neq 2$). Let $a = (a_1, a_2, \ldots a_n) \neq (0, 0, \ldots, 0)$ with $a_i \in C$ $(i = 1, 2, \ldots, n)$. Then a mapping $Q : Z \to W$ is called *quadratic with respect to a* if the functional equation

$$Q\left(\sum_{i=1}^{n} a_i z_i\right) + \sum_{1 \le i < j \le n} Q(a_j \bar{z}_i - a_i \bar{z}_j) = m \sum_{i=1}^{n} Q(z_i), \qquad (1)$$

holds for every vector $(z_1, z_2, \ldots, z_n) \in Z^n$, and a fixed real $a \neq 0$:

$$1 < m = \sum_{i=1}^{n} |a_i|^2 \neq \left[1 + \binom{n}{2}\right] \Big/ n,$$

([5]–[15]).

We note that if we set $Q(z) = |z|^2$ for $z \in C$, then the mapping $Q : C \to R$ is quadratic with respect to any $a \in C^n$, $a \neq 0$.

Also if $Q : C \to R$ is quadratic with respect to any $a \in C^n$, $a \neq 0$, then we have $Q(z) = Q(1)|z|^2$.

Finally if $Q : Z \to W$ is quadratic with respect to $a = (a_1, a_2, \ldots, a_n) \in C^n$ such that $1 < m = \sum_{i=1}^{n} |a_i|^2 \neq \left[1 + \binom{n}{2}\right] \Big/ n$, then we have

$$Q(m^n z) = (m^n)^2 Q(z), \tag{2}$$

for all $z \in Z$ and all $n \in N$.

In fact, substitution of $z_i = 0$ $(i = 1, 2, \ldots, n)$ in Eq. (1) yields

$$Q(0) + \binom{n}{2} Q(0) = m[nQ(0)],$$

or

$$\left(mn - \left[1 + \binom{n}{2}\right]\right) Q(0) = 0,$$

or

$$Q(0) = 0. \tag{1a}$$

Substituting $z_1 = z,\ z_j = 0$ $(j = 2, 3, \ldots, n)$ in Eq. (1) and employing (1a) one gets that

$$Q(a_1 z) + \sum_{j=2}^{n} Q(a_j \bar{z}) + \binom{n-1}{2} Q(0) = m[Q(z) + (n-1)Q(0)],$$

or

$$Q(a_1 z) + \sum_{j=2}^{n} Q(a_j \bar{z}) = mQ(z), \tag{2a}$$

holds for all $z \in Z$.

Moreover substitution of $z_i = \bar{a}_i z$ $(i = 1, 2, \ldots, n)$ in Eq. (1) and using (1a) one finds

$$Q(mz) + \binom{n}{2} Q(0) = m \sum_{i=1}^{n} Q(\bar{a}_i z),$$

or

$$\sum_{i=1}^{n} Q(\bar{a}_i z) = m^{-1} Q(mz), \tag{2b}$$

holds for all $z \in Z$.

Functional equations (2a)–(2b) yield

$$Q(mz) = m^2 Q(z) \tag{2c}$$

for all $z \in Z$. Then induction on $n \in N$ with z replaced by $m^{n-1}z$ yields equation (2).

Definition 2.2. Let Z and W be linear spaces (*on a field of characteristic* $\neq 2$). Let $a = (a_1, a_2, \ldots, a_n) \in C^n$ be $\neq (0, 0, \ldots, 0)$.

For $z \in Z$ and

$$1 < m = \sum_{i=1}^{n} |a_i|^2 \neq \left[1 + \binom{n}{2}\right] \Big/ n, \quad n = 2, 3, 4, \ldots,$$

set

$$Q_1^a(z) = \sum_{i=1}^{n} Q(\bar{a}_i z) \Big/ \sum_{i=1}^{n} |a_i|^2, \tag*{$(3)_1$}$$

and

$$Q_2^a(z) = \left[Q(a_1 z) + \sum_{j=2}^{n} Q(a_j \bar{z})\right] \Big/ \sum_{i=1}^{n} |a_i|^2. \tag*{$(3)_2$}$$

Then the mappings Q_i^a $(i = 1, 2) : Z \to W$ are *the quadratic weighted means of second and first form with respect to a*, respectively.

We note that if $Q : Z \to W$ is quadratic with respect to $a \in C^n : a \neq 0$, then we have *the mean functional equation with respect to a*

$$Q_1^a(z) = Q_2^a(z) \ \big(= Q(z)\big) \tag*{$[*]$}$$

for every $z \in Z$ and $i = 1, 2$, which is equivalent to the fundamental equation $(*)$.

Theorem 2.1. *Let Z be a complex normed linear space and let W be a complete normed linear space (on a field of characteristic $\neq 2$). Let $a = (a_1, a_2, \ldots, a_n) \in C^n : a \neq 0$ with $1 < m = \sum_{i=1}^{n} |a_i|^2 \neq \left[1 + \binom{n}{2}\right] \Big/ n$, for $n = 2, 3, 4, \ldots$. Assume that $f : Z \to W$ is a mapping for which there exists $\varepsilon \geq 0$ (:=a constant independent of z_i) such that the functional inequality*

$$\left\| f\left(\sum_{i=1}^{n} a_i z_i\right) + \sum_{1 \leq i < j \leq n} f(a_j \bar{z}_i - a_i \bar{z}_j) - m \sum_{i=1}^{n} f(z_i) \right\| \leq \varepsilon \tag{4$_2$}$$

holds for every vector $(z_1, z_2, \ldots, z_n) \in Z^n$. Assume in addition that $f : Z \to W$ is a mapping such that the fundamental functional inequality

$$\| f_2^a(z) - f_1^a(z) \| \leq \delta \Big/ \sum_{i=1}^{n} |a_i|^2, \tag{4$_1$}$$

holds for every complex $z \in Z$, $\delta \geq 0$ (:=a constant independent of z) and any fixed $a \neq 0$, where

$$f_2^a(z) = \left[f(a_1 z) + \sum_{j=2}^{n} f(a_j \bar{z}) \right] \Big/ \sum_{i=1}^{n} |a_i|^2, \quad \text{and} \quad f_1^a(z) = \sum_{i=1}^{n} f(\bar{a}_i z) \Big/ \sum_{i=1}^{n} |a_i|^2$$

are quadratic weighted means of second and first form, with respect to a, respectively. Then the limit

$$Q(z) = \lim_{n \to \infty} m^{-2n} f(m^n z) \tag{5}$$

exists for every $z \in Z$ and $Q : Z \to W$ is the unique quadratic mapping with respect to a such that

$$\| f(z) - Q(z) \| \leq c, \tag{6}$$

and

$$Q(z) = m^{-2n} Q(m^n z), \tag{6a}$$

hold for every complex $z \in Z$ with constant

$$c = \frac{1}{2} \frac{\left\{ 2m \left| mn - \left[1 + \binom{n}{2}\right] \right| + m(n-1)|2m - n + 2| + 2\left\{ \left| mn - \left[1 + \binom{n}{2}\right] \right| + \binom{n}{2} \right\} \right\} \varepsilon + 2m \left| mn - \left[1 + \binom{n}{2}\right] \right| \delta}{(m^2 - 1) \left| mn - \left[1 + \binom{n}{2}\right] \right|}.$$

For instance, for $n = 2$, we get $m > 1$ and

$$c = \frac{1}{2}\,\frac{(3m^2 - 1)\varepsilon + 2m(m-1)\delta}{(m-1)^2(m+1)}.$$

Note also that for $z_i = x_i \in R$ $(i = 1, 2, \ldots, n)$ we don't need inequality $(4)_1$. In this real case the above-mentioned constant c doesn't include the term $2m\left|mn - \left[1 + \binom{n}{2}\right]\right|\delta$ in the numerator.

Proof of Existence in Theorem 2.1. Substitution of $z_i = 0$ $(i = 1, 2, \ldots, n)$ in inequality $(4)_2$ yields that

$$\left\| f(0) + \sum_{1 \le i < j \le n} f(0) - m\sum_{i=1}^{n} f(0)\right\| \le \varepsilon,$$

or

$$\|f(0)\| \le \varepsilon \Big/ \left|mn - \left[1 + \binom{n}{2}\right]\right| = \begin{cases} \dfrac{\varepsilon}{mn - \left[1 + \binom{n}{2}\right]}, & m > \dfrac{1 + \binom{n}{2}}{n} \\[2ex] \dfrac{\varepsilon}{1 + \binom{n}{2} - mn}, & 1 < m < \dfrac{1 + \binom{n}{2}}{n} \end{cases}, \tag{7}$$

$1 < m \ne \left[1 + \binom{n}{2}\right]\Big/ n.$

Moreover substituting $z_1 = z,\ z_j = 0$ $(j = 2, 3, \ldots, n)$ in inequality $(4)_2$ and employing (7) as well as the triangle inequality one concludes the functional inequality

$$\left\| f(a_1 z) + \sum_{j=2}^{n} f(a_j \bar{z}) + \binom{n-1}{2} f(0) - m[f(z) + (n-1)f(0)]\right\| \le \varepsilon,$$

or

$$\left\| f_2^a(z) + \frac{1}{m}\binom{n-1}{2} f(0) - f(z) - (n-1)f(0)\right\| \le \varepsilon/m,$$

or

$$\left\| f_2^a(z) - f(z) - \frac{(n-1)(2m-n+2)}{2m} f(0) \right\| \leq \varepsilon/m,$$

or

$$\|f_2^a(z) - f(z)\| \leq \frac{\varepsilon}{m} + \frac{(n-1)|2m-n+2|}{2m} \|f(0)\|,$$

or

$$\|f_2^a(z) - f(z)\| \leq \frac{1}{2} \frac{2\left| mn - \left[1 + \binom{n}{2}\right]\right| + (n-1)|2m-n+2|}{m\left| mn - \left[1 + \binom{n}{2}\right]\right|} \varepsilon, \tag{8}$$

where

$$f_2^a(z) = \left[f(a_1 z) + \sum_{j=2}^{n} f(a_j \bar{z}) \right] \Big/ \sum_{i=1}^{n} |a_i|^2. \tag{8a}$$

In addition replacing $z_i = \bar{a}_i z$ $(i = 1, 2, \ldots, n)$ in inequality $(4)_2$ and using (7) as well as the triangle inequality, one gets the functional inequality

$$\left\| f(mz) + \binom{n}{2} f(0) - m \sum_{i=1}^{n} f(\bar{a}_i z) \right\| \leq \varepsilon,$$

or

$$\|f_1^a(z) - m^{-2} f(mz)\| \leq \frac{\varepsilon}{m^2} + \frac{1}{m^2}\binom{n}{2} \|f(0)\|,$$

or

$$\|f_1^a(z) - m^{-2} f(mz)\| \leq \frac{\left| mn - \left[1 + \binom{n}{2}\right]\right| + \binom{n}{2}}{m^2\left| mn - \left[1 + \binom{n}{2}\right]\right|} \varepsilon, \tag{9}$$

where

$$f_1^a(z) = \sum_{i=1}^{n} f(\bar{a}_i z) \Big/ \sum_{i=1}^{n} |a_i|^2. \tag{9a}$$

Moreover

$$\|f_2^a(z) - f_1^a(z)\| = \frac{\left\| f(a_1 z) + \sum_{j=2}^{n} f(a_j \bar{z}) - \sum_{i=1}^{n} f(\bar{a}_i z) \right\|}{\sum_{i=1}^{n} |a_i|^2}.$$

Employing the fundamental functional inequality $(4)_1$, one gets the equivalent inequality

$$\left\| f(a_1 z) + \sum_{j=2}^{n} f(a_j \bar{z}) - \sum_{i=1}^{n} f(\bar{a}_i z) \right\| \le \delta. \qquad (4a)_1$$

Functional inequalities (8)–(9) and $(4)_1$ (or $(4a)_1$) and the triangle inequality yield *the basic inequality*

$$\|f(z) - m^{-2}f(mz)\| \le \|f(z) - f_2^a(z)\| + \|f_2^a(z) - f_1^a(z)\| + \|f_1^a(z) - m^{-2}f(mz)\|,$$

or

$$\|f(z) - m^{-2}f(mz)\| \le c(1 - m^{-2}), \quad \text{with } 1 < m \ne \left[1 + \binom{n}{2}\right] \Big/ n, \qquad (10)$$

where c is given in Theorem 2.1.

For instance, if $n = 2 : a_1 = a_2 = 1$ or $m = 2 > 1$ and $z_i = x_i \in R$ $(i = 1, 2)$ or $\bar{z}_i = x_i \in R$ $(i = 1, 2)$, then there is *no δ-part* in c because $f_1^a(z) = f_2^a(z)$ for all real $z = x \in R$. Hence $c = (11/6)\varepsilon$. We note that in this case *a better constant* $c = (1/2)\varepsilon$ $\left(<(11/6)\varepsilon\right)$ may be found if the new substitution $z_1 = z_2 = x \in R$ is applied into inequality $(4)_2$ with $a_i = 1$ $(i = 1, 2)$. In fact, $\|f(2x) + f(0) - 4f(x)\| \le \varepsilon$ with $\|f(0)\| \le \varepsilon/2$, or $\|f(2x) - 4f(x)\| \le \varepsilon + \|f(0)\| \le \frac{3}{2}\varepsilon$, or

$$\|f(x) - 2^{-2}f(2x)\| \le c(1 - 2^{-2}), \quad c = \frac{1}{2}\varepsilon. \qquad (11)$$

By induction on $n \in N$ with z replaced by $m^{n-1}z$ in (10) we claim that *the general functional inequality*

$$\|f(z) - m^{-2n}f(m^n z)\| \le c(1 - m^{-2n}), \qquad (12)$$

holds for every complex $z \in Z$, all $n \in N$, and any fixed $a \ne 0$ such that $m > 1$ with $m \ne \left[1 + \binom{n}{2}\right] \Big/ n.$

In fact, the basic inequality (10) with z replaced by $m^{n-1}z$ yields inequality

$$\|f(m^{n-1}z) - m^{-2}f(m^n z)\| \leq c(1 - m^{-2}),$$

or

$$\|m^{-2(n-1)}f(m^{n-1}z) - m^{-2n}f(m^n z)\| \leq c(m^{-2(n-1)} - m^{-2n}), \tag{12a}$$

for all $z \in Z$.

By induction hypothesis with n replaced by $n-1$ in (12) inequality

$$\|f(z) - m^{-2(n-1)}f(m^{n-1}z)\| \leq c(1 - m^{-2(n-1)}) \tag{12b}$$

holds for all $z \in Z$.

Thus functional inequalities (12a)–(12b) and the triangle inequality imply

$$\begin{aligned}
&\|f(z) - m^{-2n}f(m^n z)\| \\
&\quad \leq \|f(z) - m^{-2(n-1)}f(m^{n-1}z)\| + \|m^{-2(n-1)}f(m^{n-1}z) - m^{-2n}f(m^n z)\| \\
&\quad \leq c[(1 - m^{-2(n-1)}) + (m^{-2(n-1)} - m^{-2n})] = c(1 - m^{-2n}),
\end{aligned}$$

completing the proof of the required inequality (12).

Claim now that the sequence $\{g_n(z)\} : g_n(z) = m^{-2n}f(m^n z)$ *converges*.

Note that from the general inequality (12) and the completeness of W one proves that the above sequence is a *Cauchy sequence*.

In fact, if $i > j > 0$, then

$$\begin{aligned}
\|g_i(z) - g_j(z)\| &= \|m^{-2i}f(m^i z) - m^{-2j}f(m^j z)\| \\
&= m^{-2j}\|m^{-2(i-j)}f(m^i z) - f(m^j z)\|,
\end{aligned} \tag{13}$$

for all complex $z \in Z$ and all $i, j \in N$.

Setting $h = m^j z$ in (13) and employing the general inequality (12) one concludes that

$$\begin{aligned}
\|g_i(z) - g_j(z)\| &= m^{-2j}\|m^{-2(i-j)}f(m^{i-j}h) - f(h)\| \\
&\leq m^{-2j}c(1 - m^{-2(i-j)}) = c(m^{-2j} - m^{-2i}) < cm^{-2j},
\end{aligned}$$

or

$$\lim_{\substack{j\to\infty \\ i>j}} \|g_i(z) - g_j(z)\| < c\left(\lim_{\substack{j\to\infty \\ i>j}} m^{-2j}\right) = 0,$$

or

$$\lim_{\substack{j\to\infty\\ i>j}} \|g_i(z) - g_j(z)\| = 0, \tag{13a}$$

completing the proof that the sequence $\{g_n(z)\}$ converges. Hence $Q = Q(z)$ is *a well-defined mapping* via the formula (5). This means that the limit (5) *exists* for all complex $z \in Z$.

In addition claim that mapping $Q : Z \to W$ *satisfies* the functional equation (1) for all complex $z_i \in Z$ $(i = 1, 2, \ldots, n)$.

In fact, it is clear from the functional inequality $(4)_2$ and the limit (5) that inequality

$$m^{-2n}\left\| f\left(\sum_{i=1}^{n} a_i m^n z_i\right) + \sum_{1\le i<j\le n} f(a_j m^n \bar{z}_i - a_i m^n \bar{z}_j) - m\sum_{i=1}^{n} f(m^n z_i)\right\| \le m^{-2n}\varepsilon, \tag{14}$$

holds for all complex $z_i \in Z$ $(i = 1, 2, \ldots, n)$, and all $n \in N$. Therefore

$$0 \le \left\| \lim_{n\to\infty} m^{-2n} f\left[m^n\left(\sum_{i=1}^{n} a_i z_i\right)\right] + \sum_{1\le i<j\le n} \lim_{n\to\infty} m^{-2n} f[m^n(a_j\bar{z}_i - a_i\bar{z}_j)] \right.$$
$$\left. - m\sum_{i=1}^{n} \lim_{n\to\infty} m^{-2n} f(m^n z_i)\right\| \le 0,$$

or

$$\left\| Q\left(\sum_{i=1}^{n} a_i z_i\right) + \sum_{1\le i<j\le n} Q(a_j\bar{z}_i - a_i\bar{z}_j) - m\sum_{i=1}^{n} Q(z_i)\right\| = 0,$$

or the mapping $Q : Z \to W$ satisfies the functional Eq. (1) for all complex $z_i \in Z$ $(i = 1, 2, \ldots, n)$. Thus Q is a quadratic mapping with respect to $a = (a_1, a_2, \ldots, a_n) \in C^n$. It is clear now from (12), $n \to \infty$, and (5) that inequality (6) holds in Z, completing *the existence proof* of Theorem 2.1.

Proof of Uniqueness in Theorem 2.1. Let $Q' : Z \to W$ be another quadratic mapping satisfying Eq. (1), such that

$$\|f(z) - Q'(z)\| \le c, \tag*{(6)$'$}$$

holds for all $z \in Z$.

If there exists a quadratic mapping $Q : Z \to W$ satisfying Eq. (1), then

$$Q' \equiv Q. \tag{15}$$

To prove the above-mentioned uniqueness (or Eq. (15)) we employ (6a) for Q and Q', as well, so that

$$Q'(z) = m^{-2n} Q'(m^n z), \tag{6a$'$}$$

holds for all $z \in Z$, and all $n \in N$. Moreover the triangle inequality and the functional inequalities (6)–(6)$'$ yield

$$\|Q(m^n z) - Q'(m^n z)\| \leq \|Q(m^n z) - f(m^n z)\| + \|f(m^n z) - Q'(m^n z)\|,$$

or

$$\|Q(m^n z) - Q'(m^n z)\| \leq 2c, \tag{16}$$

for all complex $z \in Z$, and all $n \in N$.

Then from (6a)–(6a)$'$, and (16), one proves that

$$\|Q(z) - Q'(z)\| = \|m^{-2n} Q(m^n z) - m^{-2n} Q'(m^n z)\|,$$

or

$$\|Q(z) - Q'(z)\| \leq 2(m^{-2n})c, \tag{16a}$$

holds for all $z \in Z$, and all $n \in N$.

Therefore from (16a), and $n \to \infty$, one establishes

$$0 \leq \lim_{n\to\infty} \|Q(z) - Q'(z)\| \leq 2\Big(\lim_{n\to\infty} m^{-2n}\Big)c = 0$$

or

$$\|Q(z) - Q'(z)\| = 0,$$

or

$$Q(z) = Q'(z) \tag{17}$$

for all $z \in Z$, completing the proof of *uniqueness* and thus the *stability* of Theorem 2.1.

3. Fundamental Functional Equation of Second Type

Let Z and W be linear spaces (on a field of characteristic $\neq 2$). Then we consider a non-linear mapping $Q : Z \to W$ satisfying *the fundamental functional equation of second type*

$$Q\left(\frac{a_1}{m} z\right) + \sum_{j=2}^{n} Q\left(\frac{a_j}{m} \bar{z}\right) = \sum_{i=1}^{n} Q\left(\frac{\bar{a}_i}{m} z\right), \tag{$**$}$$

with

$$0 < m = \sum_{i=1}^{n} |a_i|^2 < 1$$

and every $z \in Z$, as well as any fixed $a = (a_1, a_2, \ldots, a_n) \in C^n$, $a \neq 0$.

We note that if z_i, a_i $(i = 1, 2) \in R$, then

$$0 < m = \sum_{i=1}^{2} a_i^2 < 1,$$

and equation $(**)$ is an identity in Z.

Thus $(**)$ is *not* required.

Moreover the functional equation

$$Q(m^{-n}z) = (m^{-n})^2 Q(z) \tag{2}'$$

holds for all complex $z \in Z$, and all $n \in N$:

$$0 < m < 1.$$

In fact substitution of $z_i = 0$ $(i = 1, 2, \ldots, n)$ in Eq. (1) yields

$$Q(0) = 0. \tag{1a}'$$

Substituting $z_1 = \frac{z}{m}$, $z_j = 0$ $(j = 2, 3, \ldots, n)$ in Eq. (1) and employing (1a)$'$ one finds that

$$Q\left(\frac{a_1}{m} z\right) + \sum_{j=2}^{n} Q\left(\frac{a_j}{m} \bar{z}\right) = mQ(m^{-1}z), \tag{2a}'$$

holds for all complex $z \in Z$.

In addition substituting $z_i = \frac{\bar{a}_i}{m} z$ $(i = 1, 2, \ldots, n)$ in Eq. (1) and employing (1a)$'$ one gets that

$$\sum_{i=1}^{n} Q\left(\frac{\bar{a}_i}{m} z\right) = m^{-1} Q(z), \tag{2b}'$$

holds for all complex $z \in Z$.

Functional equations (2a)$'$–(2b)$'$ and $(**)$ yield

$$Q(m^{-1}z) = (m^{-1})^2 Q(z), \tag{2c}'$$

for all $z \in Z$. Then induction on $n \in N$ with z replaced by $m^{-(n-1)}z$ yields equation (2)$'$.

Definition 3.1. Let Z and W be linear spaces (*on a field of characteristic* $\neq 2$). Let $a = (a_1, a_2, \ldots, a_n) \in C^n$ be $\neq (0, 0, \ldots, 0)$ and $b = (b_1, b_2, \ldots, b_n) \in C^n : b_i = a_i/m$ $(i = 1, 2, \ldots, n)$. For $z \in Z$ and

$$0 < m = \sum_{i=1}^{n} |a_i|^2 < 1$$

set

$$Q_1^b(z) = \sum_{i=1}^{n} Q(\bar{b}_i z) \Big/ \sum_{i=1}^{n} |b_i|^2 = m \sum_{i=1}^{n} Q\big((\bar{a}_i/b)z\big), \tag{3)$'_1$ (}$$

and

$$\begin{aligned} Q_2^b(z) &= \left[Q(b_1 z) + \sum_{j=2}^{n} Q(b_j \bar{z}) \right] \Big/ \sum_{i=1}^{n} |b_i|^2 \\ &= m \left[Q\big((a_1/m)z\big) + \sum_{j=2}^{n} Q\big((a_j/m)\bar{z}\big) \right]. \end{aligned} \qquad (3)'_2$$

Then the mappings Q_i^b $(i = 1, 2) : Z \to W$ are *the quadratic weighted means of second and first form with respect to* b, respectively.

We note that the fundamental functional equation $(**)$ is equivalent to *the mean functional equation with respect to* b

$$Q_1^b(z) = Q_2^b(z) \ \big(= Q(z)\big) \qquad [**]$$

for every complex $z \in Z$.

Note that the functional equation $(2a)'$ comes from Eq. (2a) if we replace z by z/m. But this z-*substitution* (z by z/m) does not yield Eq. $(2)'$ directly from Eq. (2). We also note that the z-substitution or the a-*substitution* (a_i by $a_i/m : i = 1, 2$) does not yield Eq. $(2b)'$ directly from Eq. (2b). Such problems in the transition from the first section to the second section arise many times in this paper. These reasons forced us to add this second section separately.

Theorem 3.1. *Let Z be a complex normed linear space and let W be a complete normed linear space (on a field of characteristic $\neq 2$). Let $a = (a_1, a_2, \ldots, a_n) \in C^n : a \neq 0$ with $0 < m = \sum_{i=1}^{n} |a_i|^2 < 1$. Assume that $f : Z \to W$ is a mapping for which there exists $\varepsilon \geq 0$ (:=a constant independent of z_i) such that the functional inequality $(4)_2$ holds for every vector $(z_1, z_2, \ldots, z_n) \in Z^n$. Assume in addition that $f : Z \to W$ is a mapping for which the fundamental functional inequality*

$$\|f_2^b(z) - f_1^b(z)\| \leq \left(\sum_{i=1}^{n} |a_i|^2 \right) \delta', \qquad (4)'_1$$

holds for every $z \in Z$, $\delta' \geq 0$ (:=*a constant independent of* z), *and any fixed* $a \neq 0$, *where*

$$f_1^b(z) = \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} f\left((\bar{a}_i/m)z\right)\right), \quad \text{and}$$

$$f_2^b(z) = \left(\sum_{i=1}^{n} |a_i|^2\right)\left[f\left((a_1/m)z\right) + \sum_{j=2}^{n} f\left((a_j/m)\bar{z}\right)\right]$$

are quadratic weighted means of second and first form with respect to b, respectively. Then the limit

$$Q(z) = \lim_{n\to\infty} m^{2n} f(m^{-n}z) \tag{5$'$}$$

exists for every $z \in Z$ *and* $Q : Z \to W$ *is the unique quadratic mapping with respect to* $b = (b_1, b_2, \ldots, b_n) \neq 0$, *such that*

$$\|f(z) - Q(z)\| \leq c', \tag{6$''$}$$

holds for every $z \in Z$ *with constant*

$$c' = \frac{1}{2}\frac{\left\{2m\left(1+\binom{n}{2}-mn\right)+m(n-1)|n-2-2m|+2\left(1+2\binom{n}{2}-mn\right)\right\}\varepsilon+2m\left(1+\binom{n}{2}-mn\right)\delta'}{(1-m^2)\left(1+\binom{n}{2}-mn\right)}.$$

Note that for $0 < m < 1$ we have $1 + \binom{n}{2} - mn > 0$ for all $n \in N$. Take $n = 2$, then $0 < m < 1$ and

$$\|f(z) - Q(z)\| \leq \frac{1}{2}\,\frac{(3-m^2)\varepsilon + 2m(1-m)\delta'}{(1-m)^2(1+m)},$$

([13]).

To prove *the Existence in Theorem* 2.1 it is enough to prove that *the general functional inequality*

$$\|f(z) - m^{2n} f(m^{-n}z)\| \leq c'(1 - m^{2n}), \quad 0 < m < 1, \tag{12$'$}$$

holds for every $z \in Z$ and all $n \in N$.

In fact, substitution of $z_i = 0$ $(i = 1, 2, \ldots, n)$ in inequality $(4)_2$ yields

$$\|f(0)\| \leq \varepsilon \Big/ \left(1 + \binom{n}{2} - mn\right). \tag{7$'$}$$

Substituting $z_1 = z/m$, $z_j = 0$ $(j = 2, 3, \ldots, n)$ in inequality $(4)_2$ and $(7)'$ as well as the triangle inequality one gets the functional inequality

$$\left\| f\big((a_1/m)z\big) + \sum_{j=2}^{n} f\big((a_j/m)\bar{z}\big) + \binom{n-1}{2} f(0) - m[f(m^{-1}z) + (n-1)f(0)] \right\| \leq \varepsilon,$$

or

$$\|f_2^b(z) - m^2 f(m^{-1}z)\| \leq m\left\{\varepsilon + \frac{(n-1)|n-2-2m|}{2}\|f(0)\|\right\},$$

or

$$\|f_2^b(z) - m^2 f(m^{-1}z)\| \leq \frac{m}{2}\,\frac{2\left[1+\binom{n}{2} - mn\right] + (n-1)|n-2-2m|}{1+\binom{n}{2} - mn}\varepsilon, \quad (8)'$$

where $f_2^b(z)$ is the quadratic weighted mean of first form with respect to $b \neq 0 : 0 < m < 1$.

In addition replacing $z_i = (a_i/m)z$ $(i = 1, 2, \ldots, n)$ in inequality $(4)_2$ and $(7)'$ as well as the triangle inequality, one concludes the functional inequality

$$\left\| f(z) + \binom{n}{2} f(0) - m \sum_{i=1}^{n} f\big((a_i/m)z\big) \right\| \leq \varepsilon,$$

or

$$\|f_1^b(z) - f(z)\| \leq \frac{1 + 2\binom{n}{2} - mn}{1 + \binom{n}{2} - mn}\varepsilon, \qquad (9)'$$

where $f_1^b(z)$ is the quadratic weighted mean of second form with respect to $b \neq 0 : 0 < m < 1$.

Employing the fundamental functional inequality $(4)_1'$ one gets the equivalent inequality

$$\left\| f\left(\frac{a_1}{m}z\right) + \sum_{j=2}^{n} f\left(\frac{a_j}{m}\bar{z}\right) - \sum_{i=1}^{n} f\left(\frac{\bar{a}_i}{m}z\right) \right\| \leq \delta', \quad m \in (0, 1), \qquad (4\mathrm{a})_1'$$

for all complex $z \in Z$.

Functional inequalities $(8)'$–$(9)'$ and $(4)'_1$ (or $(4\text{a})'_1$) and the triangle inequality yield *the basic inequality*

$$\begin{aligned}
&\|f(z) - m^2 f(m^{-1}z)\| \\
&\quad \le \|f(z) - f_1^b(z)\| + \|f_1^b(z) - f_2^b(z)\| + \|f_2^b(z) - m^2 f(m^{-1}z)\| \\
&\quad \le \frac{2m\left[1 + \binom{n}{2} - mn\right] + m(n-1)|n-2-2m| + 2\left[1 + 2\binom{n}{2} - mn\right]}{2\left[1 + \binom{n}{2} - mn\right]}\varepsilon + m\delta',
\end{aligned}$$

or

$$\begin{aligned}
&\|f(z) - m^2 f(m^{-1}z)\| \\
&\le \frac{\left\{2m\left[1 + \binom{n}{2} - mn\right] + m(n-1)|n-2-2m| + 2\left[1 + 2\binom{n}{2} - mn\right]\right\}\varepsilon + 2m\left[1 + \binom{n}{2} - mn\right]\delta'}{2\left[1 + \binom{n}{2} - mn\right]}
\end{aligned}$$

or

$$\|f(z) - m^2 f(m^{-1}z)\| \le c'(1 - m^2), \quad 0 < m < 1 \tag{10$'$}$$

where c is given in Theorem 3.1.

For instance, if we work in R such that $z_i = x_i \in R$ $(i = 1,2)$ and $a_i = \frac{1}{2}$ $(i = 1,2)$, then $\bar{z}_i = x_i$ $(i = 1,2)$ and $m = \frac{1}{2} \in (0,1)$. In this case there is *no δ'-part* in c' because $f_1^b(z) = f_2^b(z)$ for all real $z \in R$. Hence $c' = (11/3)c$.

Note that a better constant $c' = (1/2)\varepsilon$ $\left(<(11/3)\varepsilon\right)$ may be found if new substitutions $z_i = x_i = x$ $(i = 1,2)$, $n = 2$ are applied into inequality $(4)_2$ with $a_i = \frac{1}{2}$ $(i = 1,2)$.

Replacing now z with $m^{-(n-1)}z$ in $(10)'$ one concludes that

$$\|f(m^{-(n-1)}z) - m^2 f(m^{-n}z)\| \le c'(1 - m^2),$$

or

$$\|m^{2(n-1)} f(m^{-(n-1)}z) - m^{2n} f(m^{-n}z)\| \le c'(m^{2(n-1)} - m^{2n}), \tag{12a$'$}$$

for all $z \in Z$.

By induction on $n \in N$ we claim that the general functional inequality $(12)'$ holds. By induction hypothesis with $n \to n-1$ in inequality $(12)'$

$$\|f(z) - m^{2(n-1)} f(m^{-(n-1)}z)\| \le c'(1 - m^{2(n-1)}), \tag{12b$'$}$$

holds for all $z \in Z$.

Thus the functional inequalities (12a)′–(12b)′ and the triangle inequality imply

$$\begin{aligned}&\|f(z) - m^{2n}f(m^{-n}z)\|\\&\quad\le \|f(z) - m^{2(n-1)}f(m^{-(n-1)}z)\| + \|m^{2(n-1)}f(m^{-(n-1)}z) - m^{2n}f(m^{-n}z)\|\\&\quad\le c'[(1 - m^{2(n-1)}) + (m^{2(n-1)} - m^{2n})] = c'(1 - m^{2n}),\end{aligned}$$

completing the proof of the required inequality (12)′.

The rest of the proof of Theorem 3.1 is omitted as similar to the corresponding proof of Theorem 2.1 ([1], [2], [3], [4], [16], [17]–[18]).

Example. Let us take $f : C \to C$ such that $f(z) = l|z|^2 + k$, where l, k (=complex constants): $|k| \le \varepsilon \Big/ \left[1 + \binom{n}{2} - mn\right]$, for any complex a_i $(i = 1, 2, \ldots, n) : 0 < m = \sum_{i=1}^{n} |a_i|^2 < 1$, and $n \in N : n \ge 2$.

Moreover there exists a unique quadratic mapping $Q : C \to C$ such that

$$Q(z) = \lim_{n\to\infty} m^{2n}[l|m^{-n}z|^2 + k] = l|z|^2, \quad 0 < m < 1.$$

Therefore inequality (6)″ holds. In fact, the condition on $l, k \in C$ such that

$$|k| \le \frac{\varepsilon}{1 + \binom{n}{2} - mn}, \quad m \in (0, 1) \text{ and } n \in N : n \ge 2.$$

But $f(z) = l|z|^2 + k$ satisfies $(4)_2$ if $\left|k + \binom{n}{2}k - mnk\right| \le \varepsilon$, or if

$$|k| \le \frac{\varepsilon}{1 + \binom{n}{2} - mn}.$$

Also

$$1 < \frac{2m\left(1 + \binom{n}{2} - mn\right) + m(n-1)|n - 2 - 2m| + 2\left(1 + 2\binom{n}{2} - mn\right)}{1 - m^2}, \tag{18}$$

for $0 < m < 1$ and $n \in N : n \ge 2$.

In fact, (18) is equivalent to

$$R(m, n) = 2(n-1)m^2 - 2\left[1 + \binom{n}{2} - n\right]m - 4\binom{n}{2} - m(n-1)|2m - n + 2| < 0 \tag{19}$$

for all $m \in R : 0 < m < 1$ and any $n \in N : n \ge 2$.

To prove (19) we first establish the cases: $n = 2$ and $n = 3$, separately, and then prove the general case for all $m \in R : 0 < m < 1$ and any $n \in N : n \geq 4$, as follows:

$$R(m, 2) = 2m^2 - 4 - m|2m| = -4 < 0, \quad \text{as } m \in (0, 1).$$

Also

$$R(m, 3) = 4m^2 - 2m - 12 - 2m|2m - 1| < 0,$$

because

$$\begin{aligned} R\left(m \in \left(0, \frac{1}{2}\right], 3\right) &= 4m^2 - 2m - 12 + 2m(2m - 1) \\ &= 8m^2 - 4m - 12 \\ &= 4(2m - 3)(m + 1) < 0, \quad \text{as } m \in \left(0, \frac{1}{2}\right], \end{aligned}$$

and

$$\begin{aligned} R\left(m \in \left(\frac{1}{2}, 1\right), 3\right) &= 4m^2 - 2m - 12 - 2m(2m - 1) \\ &= -12 < 0, \quad \text{as } m \in \left(\frac{1}{2}, 1\right). \end{aligned}$$

Finally claim

$$\begin{aligned} R = R\big(m \in (0, 1), n \in N : n \geq 4\big) = 2(n - 1)m^2 - 2\left[1 + \binom{n}{2} - n\right]m - 4\binom{n}{2} \\ + m(n - 1)(2m - n + 2) < 0, \end{aligned}$$

where $2m - n + 2 < 0$ or equivalently $0 < m < \frac{n}{2} - 1$ yields $m \in (0, 1)$ for $n \geq 4$ as $1 \leq \frac{n}{2} - 1$ for $n \geq 4$. This is equivalent to

$$\begin{aligned} R = {} & [2(n - 1) + 2(n - 1)]m^2 \\ & - \left\{2\left[1 + \binom{n}{2} - n\right] + (n - 1)(n - 2)\right\}m - 4\binom{n}{2} < 0, \end{aligned}$$

or

$$R_1 = R/2(n - 1) = 2m^2 - (n - 2)m - n < 0,$$

or

$$R_1 = (2m - n)(m + 1) < 0. \tag{20}$$

for all $n \in N : n \geq 4$, and $m \in (0, 1)$.

If m_1, m_2 are the roots of the equation

$$R_1 = (2m - n)(m + 1) = 0,$$

or of the equation

$$R = 2(n - 1)(2m - n)(m + 1) = 0,$$

with respect to m, then

$$m_1 = -1, \quad m_2 = n/2.$$

It is clear that

$$m \in (0, 1) \subset (m_1, m_2) = (-1, n/2),$$

for all $n \in N : n \geq 4$.

Therefore the proof of the inequality (20) is complete. Thus the inequality (18) holds.

We note that

$$\begin{aligned} &\left[1 + \binom{n}{2} - mn\right] c' \\ &\quad = \frac{1}{2} \frac{2m\left[1 + \binom{n}{2} - mn\right] + m(n-1)|n - 2 - 2m| + 2\left[1 + 2\binom{n}{2} - mn\right]}{(1 - m^2)} \\ &\qquad = \frac{(n-1)(n-2m)}{1-m} + 1 \\ &\qquad \left(= \frac{(n^2 - n + 1) - (2n - 1)m}{1 - m} > 0 \right), \end{aligned}$$

for all $n \in N : n \geq 4$ and $m \in (0, 1)$.

Therefore

$$\|f(z) - Q(z)\| = |k| \leq c', \quad 0 < m < 1.$$

We note that if $1 < m \neq \left[1 + \binom{n}{2}\right] \Big/ n$, for $n \in N : n \geq 2$, then $l \ (=Q(1)) \in C$ and any complex constant $k : |k| \leq \dfrac{\varepsilon}{\left| mn - \left[1 + \binom{n}{2}\right] \right|}$.

References

1. Aczél, J.: *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, 1966.
2. Forti, G.L.: Hyers-Ulam stability of functional equations in several variables, *Aequationes Math.* **50**, 143–190 (1995).
3. Gruber, P.M.: Stability of Isometries, *Trans. Amer. Math. Soc.* **245**, 263–277 (1978).
4. Hyers, D.H.: On the stability of the linear functional equations, *Proc. Nat. Acad. Sci.* **27**, 222–224 (1941).
5. Rassias, J.M.: On Approximation of Approximately Linear Mappings by Linear Mappings, *J. Funct. Anal.* **46**, 126–130 (1982).
6. Rassias, J.M.: On Approximation of Approximately Linear Mappings by Linear Mappings, *Bull. Sci. Math.* **108**, 445–446 (1984).
7. Rassias, J.M.: Solution of a Problem of Ulam, *J. Approx. Theory.* **57**, 268–273 (1989).
8. Rassias, J.M.: Complete Solution of the Multi-dimensional Problem of Ulam, *Discuss. Math.* **14**, 101–107 (1994).
9. Rassias, J.M.: Solution of a Stability Problem of Ulam, *Discuss. Math.* **12**, 95–103 (1992).
10. Rassias, J.M.: On the Stability of the Euler-Lagrange Functional Equation, *Chinese J. Math.* **20**, 185–190 (1992).
11. Rassias, J.M.: On the Stability of the Non-linear Euler-Lagrange Functional Equation in Real Normed Linear Spaces, *J. Math. Phys. Sci.* **28**, 231–235 (1994).
12. Rassias, J.M.: On the Stability of the Multi-dimensional Non-linear Euler-Lagrange Functional Equation, *in Geometry, Analysis and Mechanics*, pp. 275–285, World Scientific, Singapore, 1994.
13. Rassias, J.M.: On the Stability of the General Euler-Lagrange Functional Equation, *Demonstratio Math.* **29**, 755–766 (1996).
14. Rassias, J.M.: Solution of the Ulam Stability Problem for Euler-Lagrange Quadratic Mappings, *J. Math. Anal.* & *Appl.* **220**, 613–639 (1998).
15. Rassias, J.M.: Solution of the Ulam Stability Problem for 3-Dimensional Euler-Lagrange Quadratic Mappings, *Mathematica Balkanica*, (to appear), 1999.
16. Székelyhidi, L.: Note on Hyers' theorem, *C. R. Math. Rep. Acad. Sci. Canada* **8**, 127–129 (1986).
17. Ulam, S.M.: *A collection of Mathematical Problems*, Intersci. Publ., New York, 1960 (Also : Problems in modern mathematics , Wiley, New York, 1964).
18. Ulam, S.M.: *Sets, numbers and universes*, Mass. Inst. Techn. Press, Cambridge, MA. 1974.