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On the Hyers–Ulam stability problem for quadratic multi-dimensional mappings

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Summary. In 1940 S. M. Ulam proposed the well-known Ulam stability problem. In 1941 D. H. Hyers solved this problem for linear mappings. According to P. M. Gruber (1978) this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982–1999 we solved the above Ulam problem for different mappings. In this paper we solve the Hyers–Ulam stability problem for quadratic multi-dimensional mappings.

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1. Introduction

In 1940 and in 1968 S. M. Ulam [16] proposed the *general Ulam stability problem*: "When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In 1941 D. H. Hyers [4] solved this problem for linear mappings. According to P. M. Gruber [3] (1978) this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1995 G. L. Forti [2] published a survey paper which contained more information than any other publication relevant to the theory of functional equations. In 1982–1999 we ([5]–[15]) solved the above Ulam problem for different mappings. In particular, in 1996 we [13] solved the Hyers–Ulam stability problem for quadratic mappings $Q : X \to Y$ satisfying the quadratic functional equation

$$Q(a_1x_1 + a_2x_2) + Q(a_2x_1 - a_1x_2) = (a_1^2 + a_2^2)[Q(x_1) + Q(x_2)]$$

for every $x_1, x_2 \in X$, and fixed reals $a_1, a_2 \neq 0$, where X and Y are real linear spaces. In this paper we solve the Hyers–Ulam stability problem for quadratic mappings $Q : X \to Y$ satisfying the more general quadratic functional equation

$$Q\Big(\sum_{i=1}^{n} a_i x_i\Big) + \sum_{1\le i<j\le n} Q(a_j x_i - a_i x_j) = m \sum_{i=1}^{n} Q(x_i)$$

for every $x_i \in X$ $(i = 1, 2, \dots, n)$, and fixed reals $a_i \neq 0$ $(i = 1, 2, \dots, n)$, where $n \geq 2$ is arbitrary, but fixed such that

$$0 < m = \sum_{i=1}^{n} a_i^2 \neq \left[1 + \binom{n}{2}\right] \Big/ n.$$

To the best of our knowledge the afore-mentioned functional equation is established for the first time for $n > 2$.

Definition 1.1. Let X and Y be real linear spaces. Let $a = (a_1, a_2, \dots, a_n) \neq (0, 0, \dots, 0)$ with $a_i \in \mathbb{R} - \{0\}$ $(i = 1, 2, \dots, n)$, where $\mathbb{R} :=$ the set of reals. Then a mapping $Q : X \to Y$ is called *quadratic with respect to* a if the functional equation

$$Q\Big(\sum_{i=1}^{n} a_i x_i\Big) + \sum_{1\le i<j\le n} Q(a_j x_i - a_i x_j) = m \sum_{i=1}^{n} Q(x_i) \tag{1}$$

holds for every vector $(x_1, x_2, \dots, x_n) \in X^n$, and fixed real $a \neq 0$ such that

$$0 < m = \sum_{i=1}^{n} a_i^2 \neq \left[1 + \binom{n}{2}\right] \Big/ n,$$

where n is arbitrary but fixed and equals to $2, 3, 4, \dots$.

We note that if we set $Q(x) = x^2$ for $x \in \mathbb{R}$, then the mapping $Q : \mathbb{R} \to \mathbb{R}$ is quadratic with respect to any $a \in \mathbb{R}^n$, $a \neq 0$.

Definition 1.2. Let X and Y be real linear spaces, $Q : X \to Y$ be a mapping and $a = (a_1, a_2, \dots, a_n) \neq (0, 0, \dots, 0)$ with $a_i \in \mathbb{R}$ $(i = 1, 2, \dots, n)$. For $x \in X$ and $m = \sum\limits_{i=1}^{n} a_i^2 > 1 : m \neq \left[1 + \binom{n}{2}\right] / n$, where n is arbitrary but fixed and equals to $2, 3, 4, \dots$, set

$$\overline{Q}^a(x) = \sum_{i=1}^{n} Q(a_i x) \Big/ \sum_{i=1}^{n} a_i^2.$$

Then the mapping $\overline{Q}^a : X \to Y$ is called *the square of the quadratic weighted mean of* Q *with respect to* a.

For every $x \in \mathbb{R}$ set $Q(x) = x^2$. Then the mapping $\overline{Q}^a : \mathbb{R} \to \mathbb{R}$ is quadratic, such that $\overline{Q}^a(x) = x^2$. Denoting by $\sqrt{\overline{x_w^2}}$ *the quadratic weighted mean*, we note that the above-mentioned mapping $\overline{Q}^a$ is an analogous case to *the square of the quadratic weighted mean* employed in mathematical statistics: $\overline{x_w^2} = \sum_{i=1}^{n} w_i x_i^2 \Big/ \sum_{i=1}^{n} w_i$, with weights $w_i = a_i^2$, data $x_i = x$, and $Q(a_i x) = (a_i x)^2$: $i = 1, 2, \ldots, n$, where n is arbitrary but fixed and equals to $2, 3, 4, \ldots$.

If $Q : X \to Y$ is quadratic with respect to $a = (a_1, a_2, \ldots, a_n)$ such that $m = \sum_{i=1}^{n} a_i^2 > 0$ and $m \neq \left[1 + \binom{n}{2}\right]/n : n = 2, 3, 4, \ldots$, then we have

$$Q(m^p x) = (m^p)^2 Q(x) \tag{2}$$

for every $x \in X$ and any $p \in \mathbb{N}$.

In fact, substitution of $x_i = 0$ $(i = 1, 2, \ldots, n)$ in equation (1) yields

$$Q(0) + \binom{n}{2} Q(0) = mnQ(0),$$

or

$$Q(0) = 0. \tag{3}$$

For $p = 0$ (2) is trivial.

Substituting $x_1 = x$, $x_j = 0$ $(j = 2, 3, \ldots, n)$ in equation (1) and using (3) one gets that

$$Q(a_1 x) + \sum_{j=2}^{n} Q(a_j x) + \binom{n-1}{2} Q(0) = m[Q(x) + (n-1)Q(0)],$$

or

$$\overline{Q}^a(x) = Q(x) \tag{4}$$

holds for every $x \in X$.

Moreover substituting $a_i x$ $(i = 1, 2, \ldots, n)$ for x_i in equation (1) and using (3) one finds

$$Q(mx) + \binom{n}{2} Q(0) = m \sum_{i=1}^{n} Q(a_i x),$$

or

$$\overline{Q}^a(x) = m^{-2} Q(mx),$$

which holds for every $x \in X$.

From this and (4), we get

$$Q(mx) = (m)^2 Q(x).$$

Assume (2) is true. From this, with $m^p x$ in place of x, we have

$$Q(m^{p+1} x) = m^2 Q(m^p x) = m^2 (m^p)^2 Q(x) = (m^{p+1})^2 Q(x).$$

From this we prove by induction formula (2) ([1], [4]).

2. Hyers–Ulam quadratic stability

Theorem 2.1. *Let X and Y be real normed linear spaces. Assume that Y is complete. Let $0 \neq a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ be fixed with $1 < m = \sum_{i=1}^{n} a_i^2 \neq \left[1 + \binom{n}{2}\right]/n$, where n is arbitrary but fixed and equals to $2, 3, 4, \dots,$ and $0 < m < 1$. Assume that $f : X \to Y$ is a mapping for which there exists a constant $\varepsilon \geq 0$ (= a constant independent of $x_1, x_2, \dots, x_n \in X$) such that the functional inequality*

$$\left\| f\left(\sum_{i=1}^{n} a_i x_i\right) + \sum_{1 \leq i < j \leq n}^{n} f(a_j x_i - a_i x_j) - m \sum_{i=1}^{n} f(x_i) \right\| \leq \varepsilon \tag{5}$$

holds for every vector $(x_1, x_2, \dots, x_n) \in X^n$. Then the limit

$$Q(x) = \lim_{p \to \infty} \begin{cases} m^{-2p} f(m^p x), & \text{if } \ 1 < m \neq \left[1 + \binom{n}{2}\right] \Big/ n, \\ m^{2p} f(m^{-p} x), & \text{if } \ 0 < m < 1 \end{cases} \tag{6}$$

exists for every $x \in X$ and $Q : X \to Y$ is the unique quadratic mapping with respect to a, such that

$$\|f(x) - Q(x)\| \leq c \tag{7}$$

holds for every $x \in X$, where

$$c = \frac{1}{2} \frac{2m \left| mn - \left[1 + \binom{n}{2}\right] \right| + m(n-1)|2m - n + 2| + 2\left\{ \left| mn - \left[1 + \binom{n}{2}\right] \right| + \binom{n}{2} \right\}}{|m^2 - 1| \left| mn - \left[1 + \binom{n}{2}\right] \right|} \varepsilon. \tag{7a}$$

Denote $c_1 = c$, if $1 < m \neq \left[1 + \binom{n}{2}\right]/n : |m^2 - 1| = m^2 - 1$ and

$$\left| mn - \left[1 + \binom{n}{2}\right] \right| = \begin{cases} mn - \left[1 + \binom{n}{2}\right], & \text{if } m > \frac{1 + \binom{n}{2}}{n} \\ 1 + \binom{n}{2} - mn, & \text{if } 1 < m < \frac{1 + \binom{n}{2}}{n} \end{cases}.$$

Also $c_2 = c$ and if $0 < m < 1$: $|m^2 - 1| = 1 - m^2$ and $\left| mn - \left[1 + \binom{n}{2}\right] \right| = 1 + \binom{n}{2} - mn$.

Proof of Existence. Substitution of $x_i = 0$ $(i = 1, 2, \dots, n)$ in the inequality (5) yields that

$$\left\| f(0) + \sum_{1 \leq i < j \leq n} f(0) - m \sum_{i=1}^{n} f(0) \right\| \leq \varepsilon,$$

or

$$\|f(0)\| \leq \frac{\varepsilon}{\left|mn - \left[1+\binom{n}{2}\right]\right|} = \begin{cases} \frac{\varepsilon}{mn-\left[1+\binom{n}{2}\right]}, & \text{if } m > \frac{1+\binom{n}{2}}{n}, \\ \frac{\varepsilon}{1+\binom{n}{2}-mn}, & \text{if } 0 < m < \frac{1+\binom{n}{2}}{n} : m \neq 1. \end{cases} \tag{8}$$

We begin with the case $1 < m \neq \left[1+\binom{n}{2}\right]/n$ $(n = 2, 3, 4, \dots)$.

Substituting $x_1 = x$, $x_j = 0$ $(j = 2, 3, \dots, n)$ in the inequality (5) and using (8) and the triangle inequality one concludes the functional inequality

$$\left\| f(a_1 x) + \sum_{j=2}^{n} f(a_j x) + \binom{n-1}{2} f(0) - m[f(x) + (n-1)f(0)] \right\| \leq \varepsilon,$$

or

$$\left\| \overline{f}^{a}(x) + \frac{1}{m}\binom{n-1}{2} f(0) - f(x) - (n-1)f(0) \right\| \leq \frac{\varepsilon}{m},$$

or

$$\|\overline{f}^{a}(x) - f(x)\| \leq \frac{1}{2}\,\frac{2\left|mn - \left[1+\binom{n}{2}\right]\right| + (n-1)|2m - n + 2|}{m\left|mn - \left[1+\binom{n}{2}\right]\right|}\varepsilon, \tag{9}$$

where

$$\overline{f}^{a}(x) = \sum_{i=1}^{n} f(a_i x) \Big/ \sum_{i=1}^{n} a_i^2 \tag{10}$$

is *the square of the quadratic weighted mean of* f *with respect to* a.

In addition, replacing $x_i = a_i x$ $(i = 1, 2, \dots, n)$ in the inequality (5) and using (8) and (10), as well as the triangle inequality, one gets the functional inequality

$$\left\| f(mx) + \binom{n}{2} f(0) - m \sum_{i=1}^{n} f(a_i x) \right\| \leq \varepsilon,$$

or

$$\left\| \overline{f}^{a}(x) - m^{-2} f(mx) - \frac{1}{m^2}\binom{n}{2} f(0) \right\| \leq \frac{\varepsilon}{m^2},$$

or

$$\|\overline{f}^{a}(x) - m^{-2} f(mx)\| \leq \frac{\left|mn - \left[1+\binom{n}{2}\right]\right| + \binom{n}{2}}{m^2\left|mn - \left[1+\binom{n}{2}\right]\right|}\varepsilon.$$

From this, (7a) and (9), as well as the triangle inequality we get *the basic inequality*

$$\|f(x) - m^{-2} f(mx)\| \leq \|f(x) - \overline{f}^{a}(x)\| + \|\overline{f}^{a}(x) - m^{-2} f(mx)\|$$

or

$$\|f(x) - m^{-2} f(mx)\| \leq c_1(1 - m^{-2}). \tag{11}$$

By induction on $p \in \mathbb{N}$ we prove that *the general functional inequality*

$$\|f(x) - m^{-2p} f(m^p x)\| \leq c_1(1 - m^{-2p}), \tag{12}$$

holds for every $x \in X$, any $p \in \mathbb{N}$, as well as $1 < m \neq \left[1 + \binom{n}{2}\right]/n : n = 2, 3, 4, \ldots$.

The sequence $\{f_p(x)\} : f_p(x) = \{m^{-2p} f(m^p x)\}$ *converges*, because Y is complete and this sequence is clearly *a Cauchy sequence*. Hence $Q = Q(x)$ is *a well-defined mapping* via the formula (6). This means that the limit (6) exists for every $x \in X$.

In addition, we claim that mapping $Q : X \to Y$ *satisfies* the functional equation (1) for every vector $(x_1, x_2, \ldots, x_n) \in X^n$.

In fact, it is clear from the functional inequality (5) and the limit (6) that the inequality

$$m^{-2p}\left\| f\left(\sum_{i=1}^{n} a_i m^p x_i\right) + \sum_{1 \leq i < j \leq n} f(a_j m^p x_i - a_i m^p x_j) - m \sum_{i=1}^{n} f(m^p x_i)\right\| \leq m^{-2p}\varepsilon,$$

holds for every $x_i \in X$ $(i = 1, 2, \ldots, n) : n = 2, 3, 4, \ldots$, and any $p \in \mathbb{N}$. Therefore from this inequality one gets

$$\left\| Q\left(\sum_{i=1}^{n} a_i x_i\right) + \sum_{1 \leq i < j \leq n}^{n} Q(a_j x_i - a_i x_j) - m \sum_{i=1}^{n} Q(x_i)\right\| = 0,$$

thus the mapping $Q : X \to Y$ satisfies the functional equation (1) for every $x_i \in X$ $(i = 1, 2, \ldots, n)$. Thus Q is a quadratic mapping with respect to a. It is now clear from (12), (6) and letting p tend to ∞ that inequality (7) holds in X, completing *the existence proof.*

Proof of Uniqueness. Let $Q' : X \to Y$ be a quadratic mapping satisfying (7), as well as Q. Then $Q' = Q$.

To do this remember that both Q and Q' satisfy (2), too. Then from this and (2) one proves that

$$\begin{aligned}
\|Q(x) - Q'(x)\| &= \|m^{-2p} Q(m^p x) - m^{-2p} Q'(m^p x)\| \\
&\leq m^{-2n}\left\{\|Q(m^p x) - f(m^p x)\| + \|f(m^p x) - Q'(m^{px})\|\right\} \\
&\leq 2(m^{-2p}) c_1 \to 0
\end{aligned}$$

holds for every $x \in X$, and any $p \in \mathbb{N}$. Thus $p \to \infty$ completes the proof of the *stability* since $1 < m \neq \left[1 + \binom{n}{2}\right]/n$.

Consider now the case $0 < m < 1$.

We write $b_i = a_i/m$ $(i = 1, 2, \dots, n)$, $b = (b_1, b_2, \dots, b_n)$ and

$$\overline{f}^b(x) = \sum_{i=1}^{n} f(b_i x) \Big/ \sum_{i=1}^{n} b_i^2. \tag{13}$$

From (8) and (13) and the substitution of $x_1 = x/m$, $x_j = 0$ $(j = 2, 3, \dots, n)$ in inequality (5) we get

$$\left\| \sum_{i=1}^{n} f((a_i/m)x) + \binom{n-1}{2} f(0) - m\big[f(m^{-1}x) + (n-1)f(0)\big] \right\| \le \varepsilon,$$

or

$$\left\|\overline{f}^b(x) - m^2 f(m^{-1}x)\right\| \le \frac{m}{2} \frac{2\left[1 + \binom{n}{2} - mn\right] + (n-1)|n-2-2m|}{1 + \binom{n}{2} - mn}\varepsilon. \tag{14}$$

In addition, replacing x_i by $(a_i/m)x$ $(i = 1, 2, \dots, n)$ in inequality (5) and applying (8) and (13), as well as the triangle inequality, one derives the functional inequality

$$\left\| f(x) + \binom{n}{2} f(0) - m \sum_{i=1}^{n} f((a_i/m)x) \right\| \le \varepsilon,$$

or

$$\left\| \overline{f}^b(x) - f(x) - \binom{n}{2} f(0) \right\| \le \varepsilon,$$

or

$$\left\|\overline{f}^b(x) - f(x)\right\| \le \frac{1 + 2\binom{n}{2} - mn}{1 + \binom{n}{2} - mn}\varepsilon.$$

Thus from this and (14) we get *the basic inequality*

$$\left\|f(x) - m^2 f(m^{-1}x)\right\| \le \left\|f(x) - \overline{f}^b(x)\right\| + \left\|\overline{f}^b(x) - m^2 f(m^{-1}x)\right\| \le c_2(1 - m^2).$$

By induction on $p \in \mathbb{N}$ we prove that *the general functional inequality*

$$\|f(x) - m^{2p} f(m^{-p}x)\| \le c_2(1 - m^{2p}),$$

holds for every $x \in X$, any $p \in \mathbb{N}$ and $m \in (0, 1)$.

The remaining part of the proof in this case $0 < m < 1$, is omitted as similar to the corresponding proof of the previous case $1 < m \neq \left[1 + \binom{n}{2}\right]/n$.

Remarks.

1. If $n = 2$: $a_i = 1$ $(i = 1, 2)$ or $m = 2$, then $c = c_1 = \frac{11}{6}\varepsilon$.
2. If $n = 2$: $a_i = \frac{1}{2}$ $(i = 1, 2)$ or $m = \frac{1}{2}$, then $c = c_2 = \frac{11}{3}\varepsilon$.

Note that in these cases a better constant $c = \frac{1}{2}\varepsilon$ may be found if new substitutions of $x_i = x$ $(i = 1, 2)$ are applied into (5).

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