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Landau's Type Inequalities

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Let X be a complex Banach space, and let $t \to T(t)$ ($\|T(t)\| \le 1$, $t \ge 0$) be a strongly continuous contraction semigroup (on X) with infinitesimal generator A. This paper proves that

$$\|Ax\|^4 \le \frac{1024}{3}\|x\|^3\|A^4x\|, \qquad \|A^2x\|^4 \le \frac{10^4}{9}\|x\|^2\|A^4x\|^2,$$

$$\|A^3x\|^4 \le 192\|x\|\,\|A^4x\|^3$$

hold for every $x \in D(A^4)$. Inequalities are established also for uniformly bounded strongly continuous semigroups, groups, and cosine functions. © 1996 Academic Press, Inc.

1. INTRODUCTION

Edmund Landau [6] initiated the following *extremum problem*: The sharp inequality between the supremum-norms of derivatives of twice differentiable functions f such that

$$\|f'\|^2 \le 4\|f\|\,\|f''\| \tag{+}$$

holds with norm referring to the space $C[0,\infty]$.

Then R. R. Kallman and G.-C. Rota [3] found the more general result that inequality

$$\|Ax\|^2 \le 4\|x\|\,\|A^2x\| \tag{1}$$

holds for every $x \in D(A^2)$, and A the infinitesimal generator (i.e., the strong right derivative of T at zero) of $t \to T(t)$ ($t \ge 0$): a semigroup of linear contractions on a complex Banach space X

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Z. Ditzian [1] achieved the better inequality

$$\|Ax\|^2 \le 2\|x\|\,\|A^2x\| \tag{2}$$

for every $x \in D(A^2)$, where A is the infinitesimal generator of a group $t \to T(t)$ ($\|T(t)\| = 1$, $t \in \mathbb{R}$) of linear isometries on X.

Moreover H. Kraljević and S. Kurepa [4] established the even shaper inequality

$$\|Ax\|^2 \le \frac{4}{3}\|x\|\,\|A^2x\| \tag{3}$$

for every $x \in D(A^2)$, and A the infinitesimal generator (i.e., the strong right second derivative of T at zero) of $t \to T(t)$ ($t \ge 0$): a strongly continuous cosine function of linear contractions on X. Therefore the best *Landau's type constant* is $\frac{4}{3}$ (for cosine functions).

The above-mentioned inequalities (1)–(3) were extended by H. Kraljević and J. Pečarić [5] so that new *Landau's type inequalities* hold. In particular, they proved that

$$\|Ax\|^3 \le \frac{243}{8}\|x\|^2\|A^3x\|, \qquad \|A^2x\|^3 \le 24\|x\|\,\|A^3x\|^2. \tag{1$'$}$$

hold for every $x \in D(A^3)$, where A is the infinitesimal generator of a strongly continuous contraction semigroup on X, Besides they obtained the analogous but better inequalities

$$\|Ax\|^3 \le \frac{9}{8}\|x\|^2\|A^3x\|, \qquad \|A^2x\|^3 \le 3\|x\|\,\|A^3x\|^2 \tag{2$'$}$$

which hold for every $x \in D(A^3)$, where A is the infinitesimal generator of a strongly continuous contraction group on X. Moreover they got the set of analogous inequalities

$$\|Ax\|^3 \le \frac{81}{40}\|x\|^2\|A^3x\|, \qquad \|A^2x\|^3 \le \frac{72}{25}\|x\|\,\|A^3x\|^2 \tag{3$'$}$$

for every $x \in D(A^3)$, where A is the infinitesimal generator of a strongly continuous cosine function on X.

In this paper, we extend the above inequalities (1′)–(3′) so that other Landau's inequalities hold for every $x \in D(A^4)$, where A is infinitesimal generator of a uniformly bounded continuous semigroup (resp. group, or cosine function).

2. SEMIGROUPS

Let $t \to T(t)$ be uniformly bounded ($\|T(t)\| \le M < \infty$, $t \ge 0$) strongly continuous semigroup of linear operators on X with infinitesimal generator A, such that $T(0) = I$ ($:=$ Identity) in $B(X) :=$ the Banach algebra of bounded linear operators on X, $\lim_{t\downarrow 0} T(t)x = x$, for every x, and

$$Ax = \lim_{t\downarrow 0} \frac{T(t) - I}{t} x \qquad (= T'(0)x) \tag{4}$$

for every x in a linear subspace $D(A)$ ($:=$ Domain of A), dense in X [2].

For every $x \in D(A)$, we have the formula

$$T(t)x = x + \int_0^t T(u)\, Ax\, du. \tag{5}$$

Using integration by parts, we get the formula

$$\int_0^t \left(\int_0^u Tv A^2 x\, dv \right) du = \int_0^t (t-u) Tu A^2 x\, du. \tag{6}$$

Employing (6) and iterating (5), we find for every $x \in D(A^2)$ that

$$T(t)x = x + tAx + \int_0^t (t-u) Tu A^2 x\, du. \tag{5$'$}$$

Similarly iterating (5′), we obtain for every $x \in D(A^4)$ that

$$T(t)x = x + tAx + \frac{t^2}{2} A^2 x + \frac{t^3}{6} A^3 x + \frac{1}{6} \int_0^t (t-u)^3 Tu A^4 x\, du. \tag{5$''$}$$

THEOREM 1. *Let $t \to T(t)$ be a uniformly bounded* ($\|T(t)\| \le M < \infty$, $t \ge 0$) *strongly continuous semigroup of linear operators on a complex Banach space X with infinitesimal generator A, such that $A^4 x \ne 0$. Then the following inequalities*

$$\|Ax\| \le \left[M \frac{\left((ts)^2 + (sr)^2 + (rt)^2\right) + s^2(rt - sr - st)}{tsr(t-s)(s-r)} + \frac{ts + sr + rt}{tsr} \right] \|x\| + M \frac{tsr}{24} \|A^4 x\|, \tag{7}$$

$$\|A^2x\| \le 2\left[M\frac{(tr^2+sr^2+t^2s+t^2r)+s(rt-s^2)}{tsr(t-s)(s-r)}+\frac{t+s+r}{tsr}\right]\|x\| + M\frac{ts+sr+rt}{12}\|A^4x\|, \tag{7'}$$

$$\|A^3x\| \le 6\left[M\frac{(ts+sr+rt)-s^2}{tsr(t-s)(s-r)}+\frac{1}{tsr}\right]\|x\| + M\frac{t+s+r}{4}\|A^4x\|, \tag{7''}$$

hold for every $x \in D(A^4)$ *and for every* $t, s, r \in \mathbb{R}^+ = (0,\infty)$, $0 < t < s < r$.

THEOREM 2. *Let* $t \to T(t)$ *be a uniformly bounded* ($\|T(t)\| \le M < \infty$, $t \ge 0$) *strongly continuous semigroup of linear operators on a complex Banach space X with infinitesimal generator* A, *such that* $A^4x \ne 0$. *Then the following inequalities*

$$\|Ax\|^4 \le \frac{32}{81}Mg_1(m_1,m_2)\|x\|^3\,\|A^4x\|, \tag{8}$$

$$\|A^2x\|^4 \le \frac{4}{9}M^2g_2(m_1,m_2)\|x\|^2\,\|A^4x\|^2, \tag{8'}$$

$$\|A^3x\|^4 \le \frac{8}{9}M^3g_3(m_1,m_2)\|x\|\,\|A^4x\|^3, \tag{8''}$$

hold for every $x \in D(A^4)$, *and for some* $m_1, m_2 \in \mathbb{R}^+$, $m_2 > m_1 > 1$, *where*

$$g_1(m_1,m_2) = (m_1m_2) \times\left[M\frac{\left(m_1^2+(m_1m_2)^2+m_2^2\right)+m_1^2(m_2-m_1m_2-m_1)}{m_1m_2(m_1-1)(m_2-m_1)} + \frac{m_1+m_1m_2+m_2}{m_1m_2}\right]^3,$$

$$g_2(m_1,m_2) = (m_1+m_1m_2+m_2)^2 \times\left[M\frac{\left(m_2^2+m_1m_2^2+m_1+m_2\right)+m_1\left(m_2-m_1^2\right)}{m_1m_2(m_1-1)(m_2-m_1)} + \frac{1+m_1+m_2}{m_1m_2}\right]^2,$$

$$g_3(m_1,m_2) = (1+m_1+m_2)^3\left[M\frac{(m_1+m_1m_2+m_2)-m_1^2}{m_1m_2(m_1-1)(m_2-m_1)}+\frac{1}{m_1m_2}\right].$$

THEOREM 3. *Let* $t \to T(t)$ *be a strongly continuous contraction* ($\|T(t)\| \le 1$, $t \ge 0$) *semigroup of linear operators on a complex Banach space* X *with infinitesimal generator* A, *such that* $A^4 x \ne 0$. *Then the following inequalities*

$$\|Ax\|^4 \le \frac{1024}{3}\|x\|^3\,\|A^4x\|, \tag{9}$$

$$\|A^2x\|^4 \le \frac{10^4}{9}\|x\|^2\,\|A^4x\|^2, \tag{9$'$}$$

$$\|A^3x\|^4 \le 192\|x\|\,\|A^4x\|^3, \tag{9$''$}$$

hold for every $x \in D(A^4)$.

Proof of Theorem 1. In fact, formula (5″) yields the system

$$\left.\begin{aligned} 6tAx + 3t^2A^2x + t^3A^3x &= 6T(t)x - 6x - \int_0^t (t-u)^3 T(u)A^4x\,du \\ 6sAx + 3s^2A^2x + s^3A^3x &= 6T(s)x - 6x - \int_0^s (s-u)^3 T(u)A^4x\,du \\ 6rAx + 3r^2A^2x + r^3A^3x &= 6T(r)x - 6x - \int_0^r (r-u)^3 T(u)A^4x\,du. \end{aligned}\right\} \tag{10}$$

The coefficient determinant D of system (10) is

$$D = 18tsr(t-s)(s-r)(r-t). \tag{11}$$

It is clear that D is positive because of the hypothesis $0 < t < s < r$. Therefore there is a unique solution of system (10) of the form

$$Ax = \left[\frac{(sr)^2(r-s)T(t)x - (tr)^2(r-t)T(s)x + (ts)^2(s-t)T(r)x}{tsr(t-s)(s-r)(r-t)} - \frac{ts+sr+rt}{tsr}x\right] - \int_0^r K_1(t,s,r;u)T(u)A^4x\,du, \tag{12}$$

$$A^2x = 2\left[\frac{-(sr)(r^2-s^2)T(t)x + (tr)(r^2-t^2)T(s)x}{tsr(t-s)(s-r)(r-t)}\right.$$

$$-\frac{(ts)(s^2-t^2)T(r)x}{tsr(t-s)(s-r)(r-t)}$$

$$\left.+\frac{t+s+r}{tsr}x\right] + \int_0^r K_2(t,s,r;u)T(u)A^4x\,du, \qquad (12')$$

$$A^3x = 6\left[\frac{(sr)(r-s)T(t)x - (tr)(r-t)T(s)x + (ts)(s-t)T(r)x}{tsr(t-s)(s-r)(r-t)} - \right.$$

$$\left.\frac{1}{tsr}x\right] - \int_0^r K_3(t,s,r;u)T(u)A^4x\,du, \quad (12'')$$

where

$$K_1 = \begin{cases} \dfrac{(sr)^2(r-s)(t-u)^3 - (tr)^2(r-t)(s-u)^3}{6tsr(t-s)(s-r)(r-t)} \\ \qquad + \dfrac{(ts)^2(s-t)(r-u)^3}{6tsr(t-s)(s-r)(r-t)}, & 0 \le u \le t \\ \dfrac{-(tr)^2(r-t)(s-u)^3 + (ts)^2(s-t)(r-u)^3}{6tsr(t-s)(s-r)(r-t)}, & t \le u \le s, \\ \dfrac{(ts)^2(s-t)(r-u)^3}{6tsr(t-s)(s-r)(r-t)}, & s \le u \le r \end{cases}$$

$$K_2 = \begin{cases} \dfrac{(sr)(r^2-s^2)(t-u)^3 - (tr)(r^2-t^2)(s-u)^3}{3tsr(t-s)(s-r)(r-t)} \\ \qquad + \dfrac{(ts)(s^2-t^2)(r-u)^3}{3tsr(t-s)(s-r)(r-t)}, & 0 \le u \le t \\ \dfrac{-(tr)(r^2-t^2)(s-u)^3 + (ts)(s^2-t^2)(r-u)^3}{3tsr(t-s)(s-r)(r-t)}, & t \le u \le s, \\ \dfrac{(ts)(s^2-t^2)(r-u)^3}{3tsr(t-s)(s-r)(r-t)}, & s \le u \le r \end{cases}$$

$$K_3 = \begin{cases} \dfrac{(sr)(r-s)(t-u)^3 - (tr)(r-t)(s-u)^3}{tsr(t-s)(s-r)(r-t)} \\ \qquad + \dfrac{(ts)(s-t)(r-u)^3}{tsr(t-s)(s-r)(r-t)}, & 0 \le u \le t \\ \dfrac{-(tr)(r-t)(s-u)^3 + (ts)(s-t)(r-u)^3}{tsr(t-s)(s-r)(r-t)}, & t \le u \le s, \\ \dfrac{(ts)(s-t)(r-u)^3}{tsr(t-s)(s-r)(r-t)}, & s \le u \le r. \end{cases}$$

It is obvious that $K_i = K_i(t, s, r; u) \ge 0$, $i = 1, 2, 3$, for every $u \in [0, r]$ $(0 < t < s < r)$, and that the following equalities

$$\int_0^r K_1\, du = \frac{tsr}{24}, \qquad \int_0^r K_2\, du = \frac{ts + sr + rt}{12}, \qquad \int_0^r K_3\, du = \frac{t + s + r}{4}, \tag{13}$$

hold. Note that (12)–(12″) hold because the identities

$$\left.\begin{aligned} &(sr)^2(r-s) - (tr)^2(r-t) + (ts)^2(s-t) \\ &\qquad = (t-s)(s-r)(r-t)(ts + sr + rt), \\ &\qquad\quad -(sr)(r^2 - s^2) + (tr)(r^2 - t^2) - (ts)(s^2 - t^2) \\ &\qquad = -(t-s)(s-r)(r-t)(t+s+r), \\ &(sr)(r-s) - (tr)(r-t) + (ts)(s-t) \\ &\qquad = (t-s)(s-r)(r-t) \end{aligned}\right\} \tag{14}$$

hold.

Therefore from formulas (12)–(12″), (13), and the triangle inequality, we get inequalities (7)–(7″). This completes the proof of Theorem 1.

Proof of Theorem 2. Setting

$$s = m_1 t, \qquad r = m_2 t, \qquad m_2 > m_1 > 1, \qquad t > 0 \tag{15}$$

in (7)–(7″), we obtain the following inequalities

$$\|Ax\| \le a_1 \frac{1}{t} + b_1 t^3, \qquad \|A^2 x\| \le a_2 \frac{1}{t^2} + b_2 t^2, \qquad \|A^3 x\| \le a_3 \frac{1}{t^3} + b_3 t, \tag{16}$$

where

$$a_1 = \left[M \frac{\left(m_1^2 + (m_1 m_2)^2 + m_2^2\right) + m_1^2(m_2 - m_1 m_2 - m_1)}{m_1 m_2 (m_1 - 1)(m_2 - m_1)} \right.$$

$$\left. + \frac{m_1 + m_1 m_2 + m_2}{m_1 m_2} \right] \|x\|,$$

$$b_1 = M \frac{m_1 m_2}{24} \|A^4 x\|,$$

$$a_2 = 2 \left[M \frac{\left(m_2^2 + m_1 m_2^2 + m_1 + m_2\right) + m_1\left(m_2 - m_1^2\right)}{m_1 m_2 (m_1 - 1)(m_2 - m_1)} \right.$$

$$\left. + \frac{1 + m_1 + m_2}{m_1 m_2} \right] \|x\|,$$

$$b_2 = M \frac{m_1 + m_1 m_2 + m_2}{12} \|A^4 x\|,$$

$$a_3 = 6 \left[M \frac{(m_1 + m_1 m_2 + m_2) - m_1^2}{m_1 m_2 (m_1 - 1)(m_2 - m_1)} + \frac{1}{m_1 m_2} \right] \|x\|,$$

$$b_3 = M \frac{1 + m_1 + m_2}{4} \|A^4 x\|.$$

Minimizing the right-hand side functions of t of (16), we get the sharper inequalities

$$\|Ax\|^4 \le \frac{256}{27} a_1^3 b_1, \qquad \|A^2 x\|^4 \le 16 a_2^2 b_2^2, \qquad \|A^3 x\|^4 \le \frac{256}{27} a_3 b_3^3. \quad (17)$$

But

$$a_1^3 b_1 = \frac{M}{24} g_1(m_1, m_2) \|x\|^3 \, \|A^4 x\|, \qquad a_2^2 b_2^2 = \frac{M^2}{36} g_2(m_1, m_2) \|x\|^2 \, \|A^4 x\|^2,$$

and

$$a_3 b_3^3 = \frac{3M^3}{32} g_3(m_1, m_2) \|x\| \, \|A^4 x\|^3.$$

Therefore from (15)–(17), we obtain inequalities (8)–(8″). This completes the proof of Theorem 2.

Proof of Theorem 3. Taking $M = 1$, we have

$$g_1(m_1, m_2) = 8g_1^+(m_1, m_2), \qquad g_2(m_1, m_2) = 4g_2^+(m_1, m_2),$$
$$g_3(m_1, m_2) = 2g_3^+(m_1, m_2),$$

where

$$g_1^+(m_1, m_2) = (m_1 m_2)\left[\frac{m_1\left(1 + m_2 + m_2^2 - m_1 - m_1 m_2\right)}{m_2(m_1 - 1)(m_2 - m_1)}\right]^3,$$

$$g_2^+(m_1, m_2) = (m_1 + m_1 m_2 + m_2)^2\left[\frac{1 + m_2 + m_2^2 - m_1^2}{m_2(m_1 - 1)(m_2 - m_1)}\right]^2,$$

$$g_3^+(m_1, m_2) = (1 + m_1 + m_2)^3\left[\frac{1 + m_2 - m_1}{m_2(m_1 - 1)(m_2 - m_1)}\right].$$

Hence inequalities (8)–(8″) are written, as

$$\|Ax\|^4 \le \frac{256}{81} g_1^+(m_1, m_2)\|x\|^3 \, \|A^4 x\|, \tag{18}$$

$$\|A^2 x\|^4 \le \frac{16}{9} g_2^+(m_1, m_2)\|x\|^2 \, \|A^4 x\|^2, \tag{18′}$$

$$\|A^3 x\|^4 \le \frac{16}{9} g_3^+(m_1, m_2)\|x\| \, \|A^4 x\|^3, \tag{18″}$$

for some $m_1, m_2 \in \mathbb{R}^+$: $m_2 > m_1 > 1$.

All functions $g_i^+ = g_i^+(m_1, m_2)$, $i = 1, 2, 3$, attain their minimum at the same m_1, m_2: $m_1 = 2 + \sqrt{2}$, $m_2 = 3 + 2\sqrt{2}$, so that

$$\min g_1^+(m_1, m_2) = 108 = \min g_3^+(m_1, m_2), \qquad \min g_2^+(m_1, m_2) = 625. \tag{19}$$

Therefore inequalities (18)–(18″) and minima (19) yield the even sharper inequalities (9)–(9″). This completes the proof of Theorem 3.

3. GROUPS

Let $t \to T(t)$ be a uniformly bounded ($\|T(t)\| \le M < \infty$, $t \in \mathbb{R} = (-\infty, \infty)$) strongly continuous group of linear operators on X with infinitesimal generator A. It is clear that analogous inequalities (as those in the aforementioned Theorems 1–3) hold for every $t, s, r \in \mathbb{R}^- = (-\infty, 0)$, $t < s < r < 0$.

Case I. $s < 0 < t < r$. Denote

$$s = m_1 t, \qquad r = m_2 t, \qquad m_1 < 0, \qquad m_2 > 1, \qquad t > 0, \tag{15'}$$

and

$$x_1 = 6tAx, \qquad x_2 = 3t^2A^2x, \qquad x_3 = t^3A^3x, \tag{20}$$

as well as

$$a = 6T(t)x - 6x - \int_0^t (t-u)^3 T(u) A^4 x\, du,$$

$$b = 6T(m_1 t)x - 6x - \int_0^{m_1 t} (m_1 t - u)^3 T(u) A^4 x\, du$$

$$\left(= 6T(m_1 t)x - 6x - m_1^4 \int_0^t (t-u)^3 T(m_1 u) A^4 x\, du\right),$$

$$c = 6T(m_2 t)x - 6x - \int_0^{m_2 t} (m_2 t - u)^3 T(u) A^4 x\, du.$$

Then system (10) takes the form

$$x_1 + x_2 + x_3 = a, \qquad m_1 x_1 + m_1^2 x_2 + m_1^3 x_3 = b,$$
$$m_2 x_1 + m_2^2 x_2 + m_2^3 x_3 = c. \tag{10'}$$

Solving system (10′), we find the unique solution

$$x_1 = \frac{(m_1 m_2)^2 (m_2 - m_1) a - m_2^2 (m_2 - 1) b + m_1^2 (m_1 - 1) c}{m_1 m_2 (m_1 - 1)(m_2 - 1)(m_2 - m_1)} \tag{21}$$

$$x_2 = \frac{-(m_1 m_2)\left(m_2^2 - m_1^2\right) a + m_2\left(m_2^2 - 1\right) b - m_1\left(m_1^2 - 1\right) c}{m_1 m_2 (m_1 - 1)(m_2 - 1)(m_2 - m_1)} \tag{21'}$$

$$x_3 = \frac{(m_1 m_2)(m_2 - m_1) a - m_2 (m_2 - 1) b + m_1 (m_1 - 1) c}{m_1 m_2 (m_1 - 1)(m_2 - 1)(m_2 - m_1)}. \tag{21''}$$

THEOREM 4. *Let* $t \to T(t)$ *be a uniformly bounded* ($\|T(t)\| \le M < \infty$, $t \in \mathbb{R}$) *strongly continuous group of linear operators on complex Banach space*

X with infinitesimal generator A, such that $A^4x \neq 0$. *Then*

$$\|Ax\| \le \left[M\frac{(m_1m_2)^2 + m_1^2 + m_2^2 - m_1 + m_1m_2 - m_2}{m_1m_2(m_1-1)(m_2-1)} + \frac{m_1 + m_1m_2 + m_2}{m_1m_2}\right]\|x\|\frac{1}{t} + M\frac{(m_1m_2)(1+m_1-m_1m_2+m_2)}{24(m_1-1)(m_2-1)}\|A^4x\|t^3, \tag{22}$$

$$\|A^2x\| \le 2(M+1)\left(-\frac{1+m_1+m_2}{m_1m_2}\right)\|x\|\frac{1}{t^2} + \frac{M}{12}(-(m_1+m_1m_2+m_2))\|A^4x\|t^2, \tag{22$'$}$$

$$\|A^3x\| \le 6\left[M\frac{m_1+m_1m_2+m_2-m_2^2}{m_1m_2(m_2-1)(m_2-m_1)} - \frac{1}{m_1m_2}\right]\|x\|\frac{1}{t^3} + M\frac{-m_1^2m_2 - m_1m_2 + m_1m_2^2 + (m_1m_2)^2 + m_2^2 + m_2^4}{4m_2(m_2-1)(m_2-m_1)}\|A^4x\|t, \tag{22$''$}$$

hold for every $x \in D(A^4)$, *for every* $t \in \mathbb{R}^+$, *and for some* $m_1 \in \mathbb{R}^-$, $m_2 \in \mathbb{R}^+$,

$$-(m_2+1) < m_1 < -\frac{m_2}{m_2+1}, \qquad m_2 > 1.$$

THEOREM 5. *Let* $t \to T(t)$ *be a contraction* ($\|T(t)\| \le 1$, $t \in \mathbb{R}$) *strongly continuous group of linear operators on complex Banach space X with infinitesimal generator A, such that* $A^4x \neq 0$. *Then the following inequalities*

$$\|Ax\|^4 \le \frac{256}{81}f_1(m_1,m_2)\|x\|^3\|A^4x\|, \tag{23}$$

$$\|A^2x\|^4 \le \frac{16}{9}f_2(m_1,m_2)\|x\|^2\|A^4x\|^2, \tag{23$'$}$$

$$\|A^3x\|^4 \le \frac{16}{9}f_3(m_1,m_2)\|x\|\,\|A^4x\|^3, \tag{23$''$}$$

hold for every $x \in D(A^4)$, *and for some* $m_1 \in \mathbb{R}^-$, $m_2 \in \mathbb{R}^+$,

$$-(m_2+1) < m_1 < -\frac{m_2}{m_2+1}, \qquad m_2 > 1,$$

where

$$f_1(m_1, m_2) = \left(\frac{m_1 m_2}{(m_1 - 1)(m_2 - 1)} \right)^4 (1 + m_1 - m_1 m_2 + m_2),$$

$$f_2(m_1, m_2) = \left(\frac{1 + m_1 + m_2}{m_1 m_2} \right)^2 (m_1 + m_1 m_2 + m_2)^2,$$

$$f_3(m_1, m_2)$$

$$= \frac{(1 + m_1 - m_2)\left(-m_1^2 m_2 - m_1 m_2 + m_1 m_2^2 + (m_1 m_2)^2 + m_2^2 + m_2^4\right)^3}{\left(m_1 m_2^3\right)\left((m_2 - 1)(m_2 - m_1)\right)^4}.$$

THEOREM 6. *Let $t \to T(t)$ be a strongly continuous contraction ($\|T(t)\| \le 1$, $t \in \mathbb{R}$) group of linear operators on a complex Banach space X with infinitesimal generator A, such that $A^4 x \neq 0$. Then the following inequalities*

$$\|Ax\|^4 \le 10 \left(\frac{5}{6} \right)^4 \|x\|^3 \, \|A^4 x\|, \tag{24}$$

$$\|A^2 x\|^4 \le \frac{16}{9} \|x\|^2 \, \|A^4 x\|^2, \tag{24$'$}$$

$$\|A^3 x\|^4 \le 10 \frac{5^4 13^3}{2^5 3^6} \|x\| \, \|A^4 x\|^3, \tag{24$''$}$$

hold for every $x \in D(A^4)$.

THEOREM 7. *Let $t \to T(t)$ be a strongly continuous contraction ($\|T(t)\| \le 1$, $t \in \mathbb{R}$) group of linear operators on a complex Banach space X with infinitesimal generator A, such that $A^4 x \neq 0$. Then the following inequalities*

$$\|Ax\|^4 \le \frac{32}{81} \frac{m_{20}^5}{(m_{20} - 1)^4} \|x\|^3 \, \|A^4 x\|, \tag{25}$$

$$\|A^2 x\|^4 \le \frac{16}{9} \|x\|^2 \, \|A^4 x\|^2, \tag{25$'$}$$

$$\|A^3 x\|^4 \le \frac{16}{9} \frac{m_{20}^4 \left(1 + m_{20}^2\right)^3}{\left(m_{20}^2 - 1\right)^4} \|x\| \, \|A^4 x\|^3, \tag{25$''$}$$

hold for every $x \in D(A^4)$, where

$$m_{20} = \sqrt{\frac{7 + \sqrt{57}}{2}}\,.$$

Proof of Theorem 4. In fact, from (20) and (21)–(21″), we get

$$Ax = \frac{x_1}{6t}$$
$$= \left(\frac{(m_1m_2)^2(m_2 - m_1)T(t)x - m_2^2(m_2 - 1)T(m_1t)x}{m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)} \right.$$
$$+ \frac{m_1^2(m_1 - 1)T(m_2t)x}{m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)}$$
$$\left. - \frac{(m_1m_2)^2(m_2 - m_1) - m_2^2(m_2 - 1) + m_1^2(m_1 - 1)}{m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)} x \right) \frac{1}{t}$$
$$- \left[\int_0^t (t-u)^3 \left(\frac{(m_1m_2)^2(m_2 - m_1)T(u)}{6m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)} \right. \right.$$
$$- \frac{m_2^2(m_2 - 1)m_1^4T(m_1u)}{6m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)}$$
$$\left. \left. + \frac{m_1^2(m_1 - 1)m_2^4T(m_2u)}{6m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)} \right) A^4x\,du \right] \frac{1}{t}, \quad (26)$$

$$A^2x = \frac{x_2}{3t^2}$$
$$= 2\left(\frac{-m_1m_2\left(m_2^2 - m_1^2\right)T(t)x + m_2\left(m_2^2 - 1\right)T(m_1t)x}{m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)} \right.$$
$$- \frac{m_1\left(m_1^2 - 1\right)T(m_2t)x}{m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)}$$
$$\left. - \frac{-m_1m_2\left(m_2^2 - m_1^2\right) + m_2\left(m_2^2 - 1\right) - m_1\left(m_1^2 - 1\right)}{m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)} x \right) \frac{1}{t^2}$$
$$- \left[\int_0^t (t-u)^3 \left(\frac{-m_1m_2\left(m_2^2 - m_1^2\right)T(u)}{3m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)} \right. \right.$$
$$+ \frac{m_2\left(m_2^2 - 1\right)m_1^4T(m_1u)}{3m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)}$$
$$\left. \left. - \frac{m_1\left(m_1^2 - 1\right)m_2^4T(m_2u)}{3m_1m_2(m_1 - 1)(m_2 - 1)(m_2 - m_1)} \right) A^4x\,du \right] \frac{1}{t^2}, \quad (26')$$

$$A^3x = \frac{x_3}{t^3}$$
$$= 6\bigg(\frac{m_1m_2(m_2-m_1)T(t)x - m_2(m_2-1)T(m_1t)x}{m_1m_2(m_1-1)(m_2-1)(m_2-m_1)}$$
$$+\frac{m_1(m_1-1)T(m_2t)x}{m_1m_2(m_1-1)(m_2-1)(m_2-m_1)}$$
$$-\frac{m_1m_2(m_2-m_1) - m_2(m_2-1) + m_1(m_1-1)}{m_1m_2(m_1-1)(m_2-1)(m_2-m_1)}x\bigg)\frac{1}{t^3}$$
$$-\bigg[\int_0^t (t-u)^3\bigg(\frac{m_1m_2(m_2-m_1)T(u) - m_2(m_2-1)m_1^4T(m_1u)}{m_1m_2(m_1-1)(m_2-1)(m_2-m_1)}$$
$$+\frac{m_1(m_1-1)m_2^4T(m_2u)}{m_1m_2(m_1-1)(m_2-1)(m_2-m_1)}\bigg)A^4x\,du\bigg]\frac{1}{t^3}. \quad (26'')$$

But it is clear that the following identities

$$(m_1m_2)^2(m_2-m_1) - m_2^2(m_2-1) + m_1^2(m_1-1)$$
$$= (m_1-1)(m_2-1)(m_2-m_1)(m_1+m_1m_2+m_2),$$
$$-m_1m_2\left(m_2^2-m_1^2\right) + m_2\left(m_2^2-1\right) - m_1\left(m_1^2-1\right)$$
$$= -(m_1-1)(m_2-1)(m_2-m_1)(1+m_1+m_2),$$
$$-m_1m_2\left(m_2^2-m_1^2\right) + m_2\left(m_2^2-1\right)m_1^4 - m_1\left(m_1^2-1\right)m_2^4$$
$$= -m_1m_2(m_1-1)(m_2-1)(m_2-m_1)(m_1+m_1m_2+m_2)$$

hold. Applying these identities and formulas (26)–(26″), we obtain inequalities (22)–(22″). This completes the proof of Theorem 4.

Proof of Theorem 5. In fact, from inequalities (22)–(22″) we get

$$\|Ax\| \le a_1^+\frac{1}{t} + b_1^+t^3, \qquad \|A^2x\| \le a_2^+\frac{1}{t^2} + b_2^+t^2,$$
$$\|A^3x\| \le a_3^+\frac{1}{t^3} + b_3^+t, \quad (27)$$

where

$$a_1^+ = 2\frac{m_1m_2}{(m_1-1)(m_2-1)}\|x\|,$$
$$b_1^+ = \frac{(m_1m_2)(1+m_1-m_1m_2+m_2)}{24(m_1-1)(m_2-1)}\|A^4x\|,$$

$$a_2^+ = (-4)\frac{1+m_1+m_2}{m_1m_2}\|x\|,$$

$$b_2^+ = -\frac{1}{12}(m_1+m_1m_2+m_2)\|A^4x\|,$$

$$a_3^+ = 12\frac{1+m_1-m_2}{m_1(m_2-1)(m_2-m_1)}\|x\|,$$

$$b_3^+ = \frac{-m_1^2m_2 - m_1m_2 + m_1m_2^2 + (m_1m_2)^2 + m_2^2 + m_2^4}{4m_2(m_2-1)(m_2-m_1)}\|A^4x\|,$$

and new identities

$$m_1m_2(m_1-m_2) + m_2(m_2-1) + m_1(m_1-1) = (m_1-1)\left(m_1+m_1m_2+m_2-m_2^2\right),$$

$$m_1m_2(m_1-m_2) + m_2(m_2-1)m_1^4 + m_1(m_1-1)m_2^4 = m_1(m_1-1)\left(-m_1^2m_2 - m_1m_2 + m_1m_2^2 + (m_1m_2)^2 + m_2^2 + m_2^4\right).$$

Minimizing the right-hand side of (27), we find

$$\|Ax\|^4 \le \frac{256}{27}(a_1^+)^3(b_1^+), \qquad \|A^2x\|^4 \le 16(a_2^+)^2(b_2^+)^2,$$

$$\|A^3x\|^4 \le \frac{256}{27}(a_3^+)(b_3^+)^3, \tag{28}$$

where

$$(a_1^+)^3(b_1^+) = \frac{8}{24}f_1(m_1,m_2),$$

$$(a_2^+)^2(b_2^+)^2 = \frac{1}{9}f_2(m_1,m_2),$$

$$(a_3^+)(b_3^+)^3 = \frac{3}{16}f_3(m_1,m_2).$$

Therefore inequalities (28) yield inequalities (23)–(23″). This completes the proof of Theorem 5.

Proof of Theorem 6. Setting $m_1 = -1$, we get $\min f_1(-1, m_2) = 10(\frac{5}{8})^4$ at $m_2 = 5$. Hence $f_2(-1,5) = 1$, $f_3(-1,5) = 5^4(13)^3/2^9 3^4$. Therefore from formulas (23)–(23″) and $m_1 = -1$, $m_2 = 5$, we get inequalities (24)–(24″). This completes the proof of Theorem 6.

Proof of Theorem 7. In fact, $\min f_3(-1, m_2) = f_3(-1, m_{20}) = m_{20}^4(1 + m_{20}^2)^3/(m_{20}^2 - 1)^4$, where $m_{20} = \sqrt{(7 + \sqrt{57})/2}$ (= root (> 1) of equation $m_2^4 - 7m_2^2 - 2 = 0$). We have

$$f_1(-1, m_{20}) = \frac{1}{8} \frac{m_{20}^5}{(m_{20} - 1)^4}.$$

Therefore from formulas (24)–(24″), and $m_1 = -1$, $m_2 = m_{20}$, we obtain inequalities (25)–(25″). This completes the proof of Theorem 7.

Case II. $r < s < 0 < t$. Denote $s = m_1 t$, $r = m_2 t$, $m_2 = m < -1$, $m_1 = -1$, $t > 0$. From (21)–(21″), we have

$$x_1 = \frac{m^2(m+1)a - m^2(m-1)b - 2c}{2m(m^2-1)},$$

$$x_2 = \frac{a+b}{2},$$

$$x_3 = \frac{-m(m+1)a - m(m-1)b + 2c}{2m(m^2-1)}. \tag{29}$$

Therefore from (29) and (20), we obtain

$$Ax = \left(\frac{-m^2(m+1)T(t)x + m^2(m-1)T(-t)x + 2T(mt)x}{2m(1-m^2)} - \frac{1}{m}x \right)\frac{1}{t},$$
$$+ \left(\int_0^t (t-u)^3 \left[\frac{m^2(m+1)T(u) - m^2(m-1)T(-u)}{12m(1-m^2)} - \frac{2m^4 T(mu)}{12m(1-m^2)} \right] A^4 x\, du \right) \frac{1}{t}, \tag{30}$$

$$A^2x = 2\left(\frac{T(t)x + T(-t)x}{2} - x \right) \frac{1}{t^2}$$
$$- \left(\int_0^t (t-u)^3 \frac{T(u) + T(-u)}{6} A^4 x\, du \right) \frac{1}{t^2}, \tag{30′}$$

$$A^3x = 6\left(\frac{m(m+1)T(t)x + m(m-1)T(-t)x - 2T(mt)x}{2m(1-m^2)} + \frac{1}{m}x \right) \frac{1}{t^3}$$
$$+ \left(\int_0^t (t-u)^3 \left[\frac{-m(m+1)T(u) - m(m-1)T(-u)}{2m(1-m^2)} + \frac{2m^4 T(mu)}{2m(1-m^2)} \right] A^4 x\, du \right) \frac{1}{t^3}. \tag{30″}$$

Thus from formulas (30)–(30″) we get

$$\|Ax\| \leq \frac{m}{m+1}\|x\|\frac{1}{t} + \left(-\frac{m^2}{24(m+1)}\right)\|A^4x\|t^3, \tag{31}$$

$$\|A^2x\| \leq 4\|x\|\frac{1}{t^2} + \frac{1}{12}\|A^4x\|t^2, \tag{31′}$$

$$\|A^3x\| \leq 12\frac{m}{1-m^2}\|x\|\frac{1}{t^3} + \frac{1}{4}\frac{m(1+m^2)}{1-m^2}\|A^4x\|t. \tag{31″}$$

Minimizing the right-hand side of inequalities (31)–(31″), we find

$$\|Ax\|^4 \leq \frac{32}{81}h_1(m)\|x\|^3\,\|A^4x\|, \tag{32}$$

$$\|A^2x\|^4 \leq \frac{16}{9}\|x\|^2\,\|A^4x\|^2, \tag{32′}$$

$$\|A^3x\|^4 \leq \frac{16}{9}h_3(m)\|x\|\,\|A^4x\|^3, \tag{32″}$$

where

$$h_1(m) = -\frac{m^5}{(m+1)^4}, \qquad h_3(m) = \frac{m^4(1+m^2)^3}{(1-m^2)^4}, \qquad m < -1.$$

First, minimizing $h_1(m)$, $m < -1$, we get $m = -5$. Then inequalities (32)–(32″) (with $m = -5$) are the same as the inequalities (24)–(24″). Finally, minimizing $h_3(m)$, $m < -1$, we obtain $m = m_0 = -\sqrt{(7+\sqrt{57})/2}$. Then inequalities (32)–(32″) (with $m = m_0$) are the same as the inequalities (25)–(25″).

4. COSINE FUNCTIONS

Let $t \to T(t)$ $(t \geq 0)$ be a uniformly bounded $(\|T(t)\| \leq M < \infty,\ t \geq 0)$ strongly continuous cosine function with infinitesimal operator A, such that $T(0) = I$ ($:=$ identity) in $B(X)$, $\lim_{t\downarrow 0} T(t)x = x$, $\forall x$, and A is defined as the strong second derivatives of T at zero,

$$Ax = T''(0)x \tag{33}$$

for every x in a linear subspace $D(A)$, which is dense in X [5].

For every $x \in D(A)$, we have the formula

$$T(t)x = x + \int_0^t (t-u)T(u)Ax\,du. \tag{34}$$

Using integration by parts, we get from (34) the formula

$$\int_0^t (t-u)\left(\int_0^u (u-v)f(v)\,dv\right)du = \frac{1}{6}\int_0^t (t-v)^3 f(v)\,dv, \quad (35)$$

where $f(v) = TvA^2x$. Note the *Leibniz formula*:

$$\frac{d}{du}\left(\int_0^u (u-v)^n f(v)\,dv\right) = n\left(\int_0^u (u-v)^{n-1} f(v)\,dv\right). \quad (36)$$

Employing (35)–(36) and iterating (34), we find for every $x \in D(A^2)$ that

$$T(t)x = x + \frac{t^2}{2!}Ax + \frac{1}{3!}\int_0^t (t-u)^3 T(u)A^2x\,dx. \quad (34')$$

Similarly iterating (34′) we obtain for every $x \in D(A^4)$ that

$$T(t)x = x + \frac{t^2}{2!}Ax + \frac{t^4}{4!}A^2x + \frac{t^6}{6!}A^3x + \frac{1}{7!}\int_0^t (t-u)^7 T(u)A^4x\,du. \quad (34'')$$

THEOREM 8. *Let $t \to T(t)$ be a uniformly bounded* ($\|T(t) \le < M < \infty$, $t \ge 0$) *strongly continuous cosine function on a complex Banach space X with infinitesimal generator A, such that $A^4x \ne 0$. Then the following inequalities*

$$\|Ax\| \le 2\left[M\frac{(ts)^2 + (sr)^2 + (rt)^2 + s^2(rt - sr - st)}{tsr(t-s)(s-r)} + \frac{ts + sr + rt}{tsr}\right]\|x\| + M\frac{tsr}{20160}\|A^4x\|, \quad (37)$$

$$\|A^2x\| \le 24\left[M\frac{(ts^2 + sr^2 + t^2s + t^2r) + s(rt - s^2)}{tsr(t-s)(s-r)} + \frac{t + s + r}{tsr}\right]\|x\| + M\frac{ts + sr + rt}{1680}\|A^4x\|, \quad (37')$$

$$\|A^3x\| \le 720\left[M\frac{(ts+sr+rt)-s^2}{tsr(t-s)(s-r)}+\frac{1}{tsr}\right]\|x\|$$

$$+M\frac{t+s+r}{56}\|A^4x\|, \tag{37''}$$

hold for every $x \in D(A^4)$, *and for every* $t, s, r \in \mathbb{R}^+$, $0 < t < s < r$.

THEOREM 9. *Let* $t \to T(t)$ *be a uniformly bounded* ($\|T(t)\| \le M < \infty$, $t \ge 0$) *strongly continuous cosine function on a complex Banach space* X *with infinitesimal generator* A, *such that* $A^4x \ne 0$. *Then the following inequalities*

$$\|Ax\|^4 \le \frac{32}{8505}Mg_1(m_1,m_2)\|x\|^3\,\|A^4x\|, \tag{38}$$

$$\|A^2x\|^4 \le \frac{4}{1225}M^2g_2(m_1,m_2)\|x\|^2\,\|A^4x\|^2, \tag{38'}$$

$$\|A^3x\|^4 \le \frac{40}{1029}M^3g_3(m_1,m_2)\|x\|\,\|A^4x\|^3, \tag{38''}$$

hold for every $x \in D(A^4)$, *and for some* $m_1, m_2 \in \mathbb{R}^+$, $m_2 > m_1 > 1$, *where* $g_i = g_i(m_1, m_2)$ *are the same as those* g_i, $i = 1, 2, 3$, *in Theorem* 2.

THEOREM 10. *Let* $t \to T(t)$ *be a strongly continuous contraction* ($\|T(t)\| \le 1$, $t \ge 0$) *cosine function on a complex Banach space* X *with infinitesimal generator* A, *such that* $A^4x \ne 0$. *Then the following inequalities*

$$\|Ax\|^4 \le \frac{1024}{315}\|x\|^3\,\|A^4x\|, \tag{39}$$

$$\|A^2x\|^4 \le \frac{400}{49}\|x\|^2\,\|A^4x\|^2, \tag{39'}$$

$$\|A^3x\|^4 \le \frac{2880}{343}\|x\|\,\|A^4x\|^3, \tag{39''}$$

hold for every $x \in D(A^4)$.

Proof of Theorem 8. In fact, setting $t(>0)$ instead of t^2 in (34″), we get

$$T(\sqrt{t})x = x + \frac{t}{2}Ax + \frac{t^2}{24}A^2x + \frac{t^3}{720}A^3x$$

$$+\frac{1}{5040}\int_0^{\sqrt{t}}(\sqrt{t}-u)^7\,T(u)\,A^4x\,du. \tag{34'''}$$

Formula (34‴) yields

$$\begin{aligned}
&2520tAx + 210t^2A^2x + 7t^3A^3x \\
&\qquad = 5040T(\sqrt{t})x - 5040x - \int_0^{\sqrt{t}} (\sqrt{t} - u)^7 T(u) A^4x\,du \\
&2520sAx + 210s^2A^2x + 7s^3A^3x \\
&\qquad = 5040T(\sqrt{s})x - 5040x - \int_0^{\sqrt{s}} (\sqrt{s} - u)^7 T(u) A^4x\,du \qquad (10') \\
&2520rAx + 210r^2A^2x + 7r^3A^3x \\
&\qquad = 5040T(\sqrt{r})x - 5040x - \int_0^{\sqrt{r}} (\sqrt{r} - u)^7 T(u) A^4x\,du.
\end{aligned}$$

The coefficient determinant D^+ of system (10′) is

$$D^+ = 3704400tsr(t-s)(s-r)(r-t)(= D/205800). \qquad (11')$$

It is clear that $D^+ > 0$ because $0 < t < s < r$. Therefore there is a unique solution of system (10′) of the form

$$\begin{aligned}
Ax = 2\Bigg(&\frac{(sr)^2(r-s)T(\sqrt{t})x - (tr)^2(r-t)T(\sqrt{s})x}{tsr(t-s)(s-r)(r-t)} \\
&+ \frac{(ts)^2(s-t)T(\sqrt{r})x}{tsr(t-s)(s-r)(r-t)} - \frac{ts + sr + rt}{tsr}x \Bigg) \\
&- \int_0^{\sqrt{r}} K_1^+(t,s,r;u)T(u)A^4x\,du, \qquad (40)
\end{aligned}$$

$$\begin{aligned}
A^2x = 24\Bigg(&\frac{-sr(r^2-s^2)T(\sqrt{t})x + tr(r^2-t^2)T(\sqrt{s})x}{tsr(t-s)(s-r)(r-t)} \\
&- \frac{ts(s^2-t^2)T(\sqrt{r})x}{tsr(t-s)(s-r)(r-t)} + \frac{t+s+r}{tsr}x \Bigg) \\
&+ \int_0^{\sqrt{r}} K_2^+(t,s,r;u)T(u)A^4x\,du, \qquad (40')
\end{aligned}$$

$$\begin{aligned}
A^3x = 720\Bigg(&\frac{sr(r-s)T(\sqrt{t})x - tr(r-t)T(\sqrt{s})x + ts(s-t)T(\sqrt{r})x}{tsr(t-s)(s-r)(r-t)} \\
&- \frac{1}{tsr}x \Bigg) - \int_0^{\sqrt{r}} K_3^+(t,s,r;u)T(u)A^4x\,du, \qquad (40'')
\end{aligned}$$

where

$$K_1^+ = \begin{cases} \dfrac{(sr)^2(r-s)(\sqrt{t}-u)^7 - (tr)^2(r-t)(\sqrt{s}-u)^7}{2520tsr(t-s)(s-r)(r-t)} \\ \quad + \dfrac{(ts)^2(s-t)(\sqrt{r}-u)^7}{2520tsr(t-s)(s-r)(r-t)}, & 0 \le u \le \sqrt{t} \\ \dfrac{-(tr)^2(r-t)(\sqrt{s}-u)^7 + (ts)^2(s-t)(\sqrt{r}-u)^7}{2520tsr(t-s)(s-r)(r-t)}, & \sqrt{t} \le u \le \sqrt{} \\ \dfrac{(ts)^2(s-t)(\sqrt{r}-u)^7}{2520tsr(t-s)(s-r)(r-t)}, & \sqrt{s} \le u \le \sqrt{} \end{cases}$$

$$K_2^+ = \begin{cases} \dfrac{sr(r^2-s^2)(\sqrt{t}-u)^7 - (tr)(r^2-t^2)(\sqrt{s}-u)^7}{210tsr(t-s)(s-r)(r-t)} \\ \quad + \dfrac{ts(s^2-t^2)(\sqrt{r}-u)^7}{210tsr(t-s)(s-r)(r-t)}, & 0 \le u \le \sqrt{t}, \\ \dfrac{-tr(r^2-t^2)(\sqrt{s}-u)^7 + ts(s^2-t^2)(\sqrt{r}-u)^7}{210tsr(t-s)(s-r)(r-t)}, & \sqrt{t} \le u \le \sqrt{s}, \\ \dfrac{ts(s^2-t^2)(\sqrt{r}-u)^7}{210tsr(t-s)(s-r)(r-t)}, & \sqrt{s} \le u \le \sqrt{r} \end{cases}$$

$$K_3^+ = \begin{cases} \dfrac{sr(r-s)(\sqrt{t}-u)^7 - tr(r-t)(\sqrt{s}-u)^7}{7tsr(t-s)(s-r)(r-t)} \\ \quad + \dfrac{ts(s-t)(\sqrt{r}-u)^7}{7tsr(t-s)(s-r)(r-t)}, & 0 \le u \le \sqrt{t}, \\ \dfrac{-tr(r-t)(\sqrt{s}-u)^7 + ts(s-t)(\sqrt{r}-u)^7}{7tsr(t-s)(s-r)(r-t)}, & \sqrt{t} \le u \le \sqrt{s}, \\ \dfrac{ts(s-t)(\sqrt{r}-u)^7}{7tsr(t-s)(s-r)(r-t)}, & \sqrt{s} \le u \le \sqrt{r}. \end{cases}$$

Note that $K_i^+ \geq 0$, $i = 1, 2, 3$, for every $u \in [0, \sqrt{r}]$ $(0 < t < s < r)$ and

$$\int_0^{\sqrt{r}} K_1^+ \, du = \frac{tsr}{20160}, \qquad \int_0^{\sqrt{r}} K_2^+ \, du = \frac{ts + sr + rt}{1680},$$

$$\int_0^{\sqrt{r}} K_3^+ \, du = \frac{t + s + r}{56}. \tag{13'}$$

From (40)–(40″), triangle inequality, (13′), (15) and similarly as in the previous Section 2 on semigroups, we get inequalities (37)–(37″). This completes the proof of Theorem 8.

Proof of Theorem 9. From (15) and (37)–(37″) and similar calculations as in Section 2 on semigroups, we find inequalities (38)–(38″). This completes the proof of Theorem 9.

Proof of Theorem 10. Setting $M = 1$, using (38)–(38″), and minimizing $g_i(m_1, m_2)$, $i = 1, 2, 3$, as in Section 2 on semigroups, we obtain inequalities (39)–(39″). This completes the proof of Theorem 10.

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