FUNCTIONAL ANALYSIS, APPROXIMATION THEORY AND NUMERICAL ANALYSIS

ON THE EXTENDED OSTROWSKI CONSTANT

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Abstract

A. Ostrowski (1979) established that if $f(z)$ is a polynomial of degree m and $g(z)$ a polynomial of degree n, then $M_f M_g \geq M_{fg} \geq \gamma \cdot M_f M_g$, where $M_f = \max\{|f(z)| : |z| = 1\}$, and the Ostrowski constant: $\gamma = \sin^m(\pi/8m)\sin^n(\pi/8n)$. In this paper we improve γ and extend it to $f_i, i = 1, 2, \ldots, k$, in $U_r = \{z : |z| = r\}$ by applying Jensen's formula.

Theorem. *If $f_i(z) = z^{n_i} + \ldots + f_i(0)$, $f_i(0) = 1, i = 1, 2, \ldots, k$, are polynomials of degrees n_i in $D = \{z : |z| \geq 1\}$, and if the zeros (roots) a_j^i, $j = 1, 2, \ldots, n_i$, of these polynomials are such that $|a_j^i| \geq 1$, then*

$$\prod_{i=l}^{k} M_{f_i} \geq M_{\prod_{i=1}^{k} f_i} \geq \gamma_2 \prod_{i=1}^{k} M_{f_i}, \qquad (*)$$

where $\gamma_2 = 2^{-N}$, and $M_{f_i} = \max\{|f_i(z)| : |z| = 1\}$, $i = 1, 2, \ldots, k$, and

$$N = \sum_{i=1}^{k} n_i \quad \text{is the degree of } F = \prod_{i=1}^{k} f_i .$$

If $k = 2$ then $\gamma_1 = \gamma$: then Ostrowski constant [1], and our constant γ_2 is greater than γ_1, because

$$\sin\left(\frac{2}{k}\frac{\pi}{8n_i}\right) \geq \frac{2}{k}\frac{\pi}{8n_i} < \frac{1}{2} ,$$

where

$$\gamma_1 = \prod_{i=1}^{k} \sin^{n_i}\left(\frac{2}{k}\frac{\pi}{8n_i}\right) ,$$

and $i = 1, 2, \ldots, k$, and $k = 2, 3, \ldots$. Assume $M_{f_i}^r = \max\{|f_i(z)| : |z| = r\}$, $i = 1, 2, \ldots, k$, and

$$\gamma_2^r = (2r)^{-N} = \gamma_2 r^{-N} .$$

If f_i, $i = 1, 2, \ldots, k$, are polynomials of degrees n_i, $i = 1, 2, \ldots, k$, in U_r, then $(*)$ is extended to the following form

$$\prod_{i=1}^{k} M_{f_i}^r \geq M_F^r \geq \gamma_2^r \cdot \prod_{i=1}^{k} M_{f_i}^r . \qquad (**)$$

Proof of Theorem. It is clear that the left hand side relation of $(*)$ holds. To prove the right hand side relation of $(*)$ we assume

$$f_i(z) = z^{n_i} + \ldots + f_i(0), \quad f_i(0) = 1 ,$$

are polynomials of degrees n_i in $D = \{z : |z| \geq 1\}$, $U = \partial D = \{z : |z| = 1\}$. In fact,

$$f_i(z) = \prod_{j=1}^{n_i} \left(z - a_j^i\right) , \qquad i = 1, 2, \ldots, k ,$$

or

$$|f_i(z)| \leq \prod_{j=1}^{ni} (1 + |a_j^i|)$$

in U, $i = 1, 2, \ldots, k$, where a_j^i, $j = 1, 2, \ldots, n_i$, are zeros of f_i in D. Therefore

$$|a_j^i| \geq 1, \text{ or } 1 + |a_j^i| \leq 2|q_j^i| .$$

Hence

$$M_{f_i} \leq 2^{n_i} \prod_{j=1}^{n_i} |a_j^i|, \qquad i = 1, 2, \ldots, k ,$$

or

$$\prod_{i=1}^{k} M_{f_i} \le 2^N \prod_{i=1}^{k}\prod_{j=1}^{n_i} |a_j^i|$$
$$= 2^N e^{\sum_{i=1}^{k}\sum_{j=1}^{n_i} \ln|a_j^i|} .$$

By Jensen's formula ([2], p. 128, and p. 139) we get

$$\sum_{j=1}^{n_i} \ln|a_j^i| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f_i(e^{it})|dt .$$

Applying this formula we get

$$\prod_{i=1}^{k} M_{f_i} \le 2^{\sum_{i=1}^{k} n_i} e^{\frac{1}{2\pi}\int_0^{2\pi} \ln|\prod_{i=1}^{k} f_i(e^{it})|dt} ,$$

or

$$\prod_{i=1}^{k} M_{f_i} \le \gamma_2^{-1} M_F ,$$

completing the proof of the Theorem.

Similarly, we prove (**). In fact, we employ the extended Jensen's formula ([2], p. 128, and p. 139)

$$\sum_{j=1}^{n_i} \ln|a_j^i/r| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f_i(re^{it})|dt ,$$

and assume polynomials f_i, $i = 1, 2, \dots, k$, in $D_r = \{z : |z| \ge r\}$. Thus

$$|f_i(z)| \le r^{n_i} \prod_{j=1}^{n_i} (1 + |a_j^i/r|)$$

in $U_r = \partial D_r = \{z : |z| = r\}$, where a_j^i are zeros of f_i in D_r. The rest of the proof is omitted as analogous to the one of the above Theorem.

References

1. A. M. Ostrowski, *Notiz über Maximalwerte von Polynomen auf dem Einheitskreis*, Univ. Beograd, Publ. Elektrotehn, Fak., Sek. Mat. Fiz. 634–677, 1979, 55–56.
2. G. Polya and G. Szegö, *Problems and Theorems in Analysis*, Springer Study Edition, Transl. (in English), Vol. I, 1972, p. 128 and p. 139.