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ON THE WELL-POSED TRICOMI PROBLEM IN $\mathbb{R}^2$

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Abstract

S. A. Chaplygin (1904) considered a two dimensional adiabatic potentional flow of a perfect gas. Then the equation of this flow was transformed to a linear equation of mixed type. F. G. Tricomi (1923), F. I. Frankl (1945), S. Nocilla (1947), C. Ferrari (1959) and most of the recent workers in the field of mixed type problems have restricted their attention to the Chaplygin equation: $K(y)u_{xx} + u_{yy} = 0$ and considered equation : $K(y)u_{xx} + u_{yy} + r(x, y)u = f(x,y)$ because of the difficulties that arise when $r \neq 0$. In 1988 (J. Math. Phys. Sci., 681–695) we established uniqueness results for (quasi -) regular solutions for the Tricomi problem even with $r \neq 0$. In this paper we consider the Tricomi problem with $r \neq 0$, impose a new a, b, c – method, and prove well-posedness in the sense that there is at most one quasi-regular solution and a weak solution exist. This problem is interesting in trasonic aerodynamic.

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THE TRICOMI PROBLEM

Consider

$$Lu = K(y)u_{xx} + u_{yy} + r(x, y)u = f(x,y), \qquad (*)$$

$$K \in C^2(\cdot),\ r \in C^1(\cdot),\ f \in C^0(\cdot),\ K(y) \gtreqless 0 \text{ whenever } y \gtreqless 0.$$

Consider a mixed domain D which is a simply–connected region and contains the parabolic arc of degeneracy $A'A$ with endpoints $A' = (-1,0)$, $A = (1,0)$ and has boundary $G = \partial D = g_1 \cup g_2 \cup g_3$ with boundary curves: g_1 a piece-wise smooth "elliptic arc" for $y > 0$ connecting points A', A and g_2

a characteristic for $y < 0$ emanating from point A and g_3 a characteristic for $y < 0$ emanating from point A':

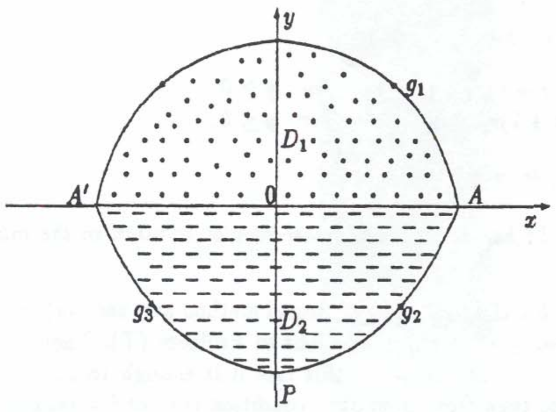
$$g_2 : x = \int_0^y \sqrt{-K(t)}\, dt + 1, \quad g_3 : x = -\int_0^y \sqrt{-K(t)}\, dt - 1.$$

Denote by D^+: "elliptic region" : $= \{(x,y) \in D,\ -1 < x < 1,\ y > 0\}$, D^-: "hyperbolic region" : $= \{(x,y) \in D,\ -1 < x < 1,\ y < 0\}$.

The above two characteristic curves interesect at the point $P = g_2 \cap g_3$ in D^- such that $P = (0, y_p)$ and

$$\int_0^{y_p} \sqrt{-K(t)}\, dt = -1.$$

Note that if $K = y$ (Tricomi's equation), then $y_p = -(\frac{3}{2})^{\frac{2}{3}}$.



Assume condition

$$u = 0 \text{ on } g_1 \cup g_2. \tag{**}$$

The Tricomi problem, or *Problem* (T): consists in finding a function $u = u(x,y)$ which satisfies equation $(*)$ in D and boundary condition $(**)$.

1. Quasi–regular Solutions

Definition 1. A function u is a *quasi–regular solution* of Problem (T) if

(i) $u \in C^2(D) \cap C(\overline{D})$, $\overline{D} = D \cup G$,

(ii) Green's theorem is applicable to the integrals $\int_D \int uLudxdy$, $\int_D \int u_x Ludxdy$, $\int_D \int u_y Ludxdy$,

(iii) the boundary and region (surface) itegrals which arise exist,

(iv) u satisfies equation $(*)$ in D and boundary condition $(**)$.

Uniqueness Theorem: *Assume the afore–mentioned domain $D \subset \mathbb{R}^2$, and conditions:*

(R_1) $\quad r \leq 0$ *on* g_3,

(R_2) $\quad (x+1)dy - ydx \geq 0$ *on* g_1,

(R_3) $\quad \begin{cases} 2r + (x+1)r_x + yr_y \leq 0 & \textit{for} \quad y \geq 0, \\ r + (x+1)r_x \leq 0 & \textit{for} \quad y \leq 0, \end{cases}$

(R_4) $\quad K' > 0$ *for* $y \geq 0$.

Then Problem (T) *has at most one quasi–regular solution in the mixed domain D.*

Proof. We apply the b, c classical energy integral method and use $(**)$. First, we assume u_1, u_2 two quasi–regular solutions of Problem (T). Then claim that $u = u_1 - u_2 = 0$ in D. To prove this fact it is enough to show that $u = 0$ on g_3, because then from boundary condition $(**)$ and a maximum principle we get $u = 0$ in D.

Then investigate

$$0 = J = 2 \int \int_D (bu_x + cu_y)Ludxdy, \quad (Lu = Lu_1 - Lu_2 = 0 \text{ in } D)$$

where

$$(c) \qquad b = x + 1 \text{ in } D, \; c = \begin{cases} y, & y \geq 0 \\ 0, & y \leq 0. \end{cases}$$

Consider the identities

$2bruu_x = (bru^2)_x - (br)_x u^2,\ \ 2cruu_y = (cru^2)_y - (cr)_y u^2,$

$2bKu_x u_{xx} = (bKu_x^2)_x - b_x K u_x^2,\ \ 2bu_x u_{yy} = (2bu_x u_y)_y - (bu_y^2)_x + b_x u_y^2,$

$2cKu_y u_{xx} = (2cKu_x u_y)_x - (cKu_x^2)_y + (cK)_y u_x^2,\ \ 2cu_y u_{yy} = (cu_y^2)_y - c_y u_y^2.$

Then employing above identities and applying Green's theorem, we obtain $0 = J = I_1 + I_2 + J_1 + J_3$, where

$$I_1 = -\int\int_D [(br)_x + (cr)_y]u^2 dxdy,\ \ I_2 = \int\int_D [(-b_x K + (cK)_y)u_x^2 +$$

$$+(b_x - c_y)u_y^2]dxdy,\ \ J_1 = \oint_G (bv_1 + cv_2)ru^2 ds,$$

$$J_3 = \oint_G Q(u_x, u_y)ds,$$

and $v = (v_1, v_2) = (dy/ds, -dx/ds)$ is outer unit normal vector on G, and

$$Q = Q(u_x, u_y) = (bv_1 - cv_2)Ku_x^2 + 2(bv_2 + cKv_1)u_x u_y + (-bv_1 + cv_2)u_y^2.$$

Claim that all integrals I_1, I_2, J_1 and J_3 are : ≥ 0.

First, the integrals I_1, I_2 *are non–negative if the following conditions* hold in D:

$$(b_x + c_y)r + (br_x + cr_y) \leq 0,\ \ -b_x K + (cK)_y \geq 0,\ \ b_x - c_y \geq 0.$$

In fact, for $y \geq 0$: $2r + (x+1)r_x + yr_y \leq 0$ holds from (R_3), $-b_x K + (cK)_y = yK' \geq 0$ holds from (R_4). Also, for $y \leq 0$: $r + (x+1)r_x \leq 0$ holds from (R_3).

Second, the integrals J_1, J_3 *are non - negative if the following conditions* hold on G:

$$r \leq 0 \text{ on } g_3 \text{ (from } (R_1)), \text{ and } bdy - cdx \geq 0 \text{ on } g_1.$$

In fact,

$$J_1 = \int_{g_1} + \int_{g_2} + \int_{g_3} = \int_{g_3} (x+1)v_1 ru^2 ds$$

(because of (B)), or

$$J_1 \geq 0, \text{ because of } (R_1), \text{ and the geometry of } g_3.$$

Similarly,

$$J_3 = \int_{g_1} + \int_{g_2} + \int_{g_3} = \int_{g_3} [(x+1)v_1 + yv_2] H N^2 ds$$

$$+ \int_{g_2} (x+1) v_1 H N^2 ds + \int_{g_3} Q(u_x, u_y) ds,$$

where N: normalizing factor: $u_x = Nv_1$, $u_y = Nv_2$ on $g_1 \cup g_2$, and $H = Kv_1^2 + v_2^2 = 0$ on $g_2 \cup g_3$, and $H > 0$ on g_1. Then, $J_3 \geq 0$ because of (R_2), and $Q \geq 0$ on g_3 because of

$$Q|_{g_3} = (\sqrt{-K}u_x - u_y)^2 (x+1)(-v_1)|_{g_3} \geq 0$$

and

$$v_1|g_3 < 0, \ \ (x+1)|_{g_3} = -\int_0^y \sqrt{-K(t)}dt > 0.$$

2. Weak Solutions

Denote $\alpha = (\alpha_1, \alpha_2)$; $\alpha_1, \alpha_2 :\geq 0$, $|\alpha| = \alpha_1 + \alpha_2$, $p = (x, y) \in \mathbb{R}^2$, $q = (\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$, and $p^\alpha = x^{\alpha_1} y^{\alpha_2}$, $q = \widetilde{x}^{\alpha_1} \widetilde{y}^{\alpha_2}$, $< p, q >= x\widetilde{x} + y\widetilde{y}$, $|p| =< p, p >^{\frac{1}{2}}$. also denote $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$, $(\partial^\alpha u)(p) = (\partial_1^{\alpha_1} \partial_2^{\alpha_2} u)(p)$ for sufficiently smooth functions $u = u(p)$. Consider adjoint equation

$$L^+ w = K(y) w_{xx} + w_{yy} + r(x, y) w = f(x, y), \qquad [*]$$

where $L^+ (= L)$ is the formal adjoint operator of the formal operator L. Note in general that if

$$Lu = \sum_{i,j=1}^{2} \alpha_{ij}(p) \partial_i \partial_j u + \sum_{i=1}^{2} \alpha_i(p) \partial_i u + \alpha(p) u,$$

then

$$L^+ w = \sum_{i,j=1}^{2} \partial_i \partial_j (\alpha_{ij}(p) w) - \sum_{i=1}^{2} \partial_i (\alpha_i(p) w) + \alpha(p) w.$$

Assume adjoint boundary conditions

$$w = 0 \ \text{ on } \ g_1 \cup g_3 \qquad [**]$$

Denote by

$$W_2^m(D) = W^{m,2}(D) = \{u(p) : p \in D,\ \partial^\alpha u(p) \in L^2(D),\ |\alpha| \le m\}$$

the Sobolev space with norm

$$\|u\|_m = \left(\sum_{|\alpha| \le m} \|\partial^\alpha u\|^2_{L^2(D)} \right)^{\frac{1}{2}} .$$

Besides denote

- $D(L) = \{u \in C^2 : u = 0 \text{ on } g_1 \cup g_2\}$: domain of the formal operator L,
- $W_2^m(D, bd) = \overline{D(L)}|_{\|\cdot\|_m}$: closure of $D(L)$ with norm $\|\cdot\|_m$,
- $D(L^+) = \{w \in C^2(\overline{D}) : w = 0 \text{ on } g_1 \cup g_3\}$: domain of the adjoint L^+,
- $W_2^m(D, bd^+) = \overline{D(L^+)}|_{\|\cdot\|_m}$: closure of $D(L^+)$ with norm $\|\cdot\|_m$,

or equivalently

$$W_2^m(D, bd^+) = \{w \in W_2^m(D) :< Lu, w >_0 =< u, L^+ w >_0$$

$$\text{for all } u \in W_2^m(D, bd)\}.$$

Definition 2. A function $u = u(p) \in L^2(D)$ is a *weak solution* of Problem (T) if $< f, w >_0 =< u, L^+ w >_0$ for all w in the space $W_2^2(D, bd^+)$.

Criterion ([1]). A necessary and sufficient condition for the existence of a weak solution of Problem (T) is that the following a priori estimate

(AP) $\quad \|w\|_0 \le C\|L^+ w\|_0,\ C = \text{const.} > 0,$

holds for all $w \in W_2^2(D, bd^+)$, or

$[AP]$ $\quad \|w\|_1 \le C\|L^+ w\|_0,\ C = \text{const.} > 0.$

where

$$\|w\|_0^2 = \iint_D w^2 dx dy \le \iint_D (w^2 + w_x^2 + w_y^2) dx dy = \|w\|_1^2.$$

Existence Theorem: *Assume the afore-mentioned domain $D \subset \mathbb{R}^2$, and the conditions*

$$[R_1] \quad \begin{cases} r \leq 0 & on \quad g_2 \\ K' > 0 & on \quad g_2, \end{cases}$$

$$[R_2] \quad (x - c_1)dy - (y + c_2)dx \geq 0 \ on \ g_1,$$

$$[R_3] \quad \begin{cases} 3r + (x - c_1)r_x + (y + c_2)r_y + \frac{1}{4}m_1 \leq -d_1^1 < 0 & for \quad y \geq 0 \\ 2r + (x - c_1)r_x + c_2 r_y + \frac{1}{4}m_1 < -d_1^2 < 0 & for \quad y \leq 0, \end{cases}$$

$$[R_4] \quad \begin{cases} K + (y + c_2)K' - m_2(x - c_1)^2 \geq d_2^1 > 0 & for \quad y \geq 0 \\ c_2 K' - m_2(x - c_1)^2 \geq d_2^2 > 0 & for \quad y \leq 0, \end{cases}$$

$$[R_5] \quad \begin{cases} 1 - m_3(y + c_2)^2 \geq d_3^1 > 0 & for \quad y \geq 0 \\ 2 - m_3(c_2)^2 \geq d_3^2 > 0 & for \quad y \leq 0, \end{cases}$$

$$[R_6] \quad \int_0^y \sqrt{-K(t)}dt + c_2\sqrt{-K(y)} - c_0 \leq 0 \ on \ g_2,$$

where $c_1 = 1 + c_0$, and $c_0, c_2, m_i, i = 1,2,3$: are positive constants. Then, the a priori estimate : $\|w\|_1 \leq C\|L^+\|_0$, $C = \text{const.} > 0$ holds and a weak solution of Problem (T) in D exists.

Proof. We apply the a^+, b^+, c^+ energy integral method and use adjoint conditions [**]. Denote $M^+w = a^+w + b^+w_x + c^+w_y$, where

$$[C] \qquad a^+ = -\frac{1}{2}, \ b^+ = x - c_1 \text{ in } D, \ c^+ = \begin{cases} y + c_2 & \text{for} \quad y \geq 0 \\ c_2 & \text{for} \quad y \leq 0 \end{cases}.$$

Then, emploing Green's theorem we get
$J^+ = 2 < M^+w, L^+w >_0 = \iint_D 2M^+wL^+w dxdy = I_1^+ + I_2^+ + J_1^+ + J_2^+ + J_3^+$,
where
$I_1^+ = \iint_D A_1^+ w^2 dxdy$, $I_2^+ = \iint_D (A_2^+ w_x^2 - 2B^+ w_x w_y + A_3^+ w_y^2) dxdy$,
$J_1^+ = \oint_G B_1^+ w2ds$, $J_2^+ = \oint_G B_2^+ ds$, $J_3^+ = \oint_G Q^+(w_x, w_y)ds$,
such that
$A_1^+ = (2a^+ - b_x^+ - c_y^+)r - (b^+ r_x + c^+ r_y) + Ka_{xx} + a_{yy}^+$, $A_2^+ = -2a^+K - b_x^+K + (c^+K)_y$,
$A_3^+ = -2a^+ + b_x^+ - c_y^+$, $B^+ = b_y^+ + c_x^+K$ (from $[C]$: $B^+ = 0$), $B_1^+ = (b^+v_1 + c^+v_2)r$,
$B_2^+ = 2a^+w(Kw_xv_1 + w_yv_2) - (Ka_x^+v_1 + a_y^+v_2)w^2$, and
$Q^+ = (b^+v_1 - c^+v_2)Kw_x^+ + 2(b^+v_2 + c^+Kv_1)w_xw_y + (-b^+v_1 + c^+v_2)w_y^+$.

From $(\sqrt{m_i}|a| - \frac{1}{\sqrt{m_i}}|b|)^2 \geq 0$, $m_i > 0$, $i = 1,2,3$, we get $m_i a^2 + \frac{1}{\sqrt{m_i}} b^2 \geq 2|ab|$, $m_i > 0$, $i = 1,2,3$.

Therefore

$$J^+ \leq \int\int_D [m_1(a^+)^2 w^2 + m_2(b^+)^2(w_x)^2 + m_3(c^+)^2(w_y)^2] dxdy + C_1^+ \|L^+ w\|_0^2,$$

where $C_1 = (\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3})^{\frac{1}{2}}$. But it is clear from $[R_1]-[R_2]$, and $[R_6]$ that $J_1^+, J_2^+, J_3^+ \geq 0$, or $J^+ \geq I_1^+ + I_2^+$.

Thus

$I_1^+ + I_2^+ \leq \int\int_D [m_1(a^+)^2 w^2 + m_2(b^+)^2 w_x^+ + m_3(c^+)^2 w_y^2] dxdy + C_1^2 \|L^+ w\|_0^2$, or $\int\int_D e_1 w^2 dxdy + \int\int_D (e_2 w_x^2 + e_3 w_y^2) dxdy \leq C_1^2 \|L^+ w\|_0^2$, where $e_1 = A_1^+ - m_1(a^+)^2$, $e_2 = A_2^+ - m_2(b^+)^2$, $e_3 = A_3^+ - m_3(c^+)^2$.

It is obvious from conditions $[R_3]-[R_5]$ that

$$C_2^2 \int\int_D (w^2 + w_x^2 + w_y^2) dxdy \leq C_1^2 \|L^+ w\|_0^2; \ \ C_1, C_2 : \text{const.} > 0$$

or

$$\|w\|_1 \leq C \|L^+\|_0, \ \ C = C_1/C_2 : \text{const.} > 0,$$

It completes the prof of the a priori estimate $[AP]$ and by Criterion [1] the the Existence Theorem also follows.

It is worth nothing that the above proof: $J_1^+, J_2^+, J_3^+ \geq 0$. To prove this it is enough to show that the the following hold on G $b^+ dy - c^+ dx \geq 0$ on g_1, $(b^+ - c^+\sqrt{-K})r \geq 0$ on g_2, $b^+ + c^+\sqrt{-K} \leq 0$ on g_2, $a_x^+\sqrt{-K} + a_y^+ + (a^+ K'/4K) \geq 0$ on g_2. In fact, these conditions hold because of the geometry of g_1, g_2, and conditions $[R_1]-[R_2]$, and $[R_6]$.

Therefore, we have proved the following new

Theorem. *If conditions* $(R_1)-(R_4)$, *and* $[R_1]-[R_6]$ *hold, then Problem* (T) *is well-posed in* D *in the sense that : There is at most one quasi-regular solution and a weak solution exists.*

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