THE BI-HYPERBOLIC DEGENERATE BOUNDARY VALUE PROBLEM IN $\mathbf{R}^3$

by

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Introduction

In this paper we impose a new boundary value problem in the theory of mixed type partial differential equations, the bi-hyperbolic degenerate problem, and get certain uniqueness results for this problem by applying the energy integral method. We consider the equation:

$$Lu = k(z)(u_{xx}-u_{yy})+u_{zz}+\lambda(x, y, z)\, u = f(x, y, z) \tag{1}$$

($k(z) \gtrless 0$ whenever $z \gtrless 0$) in a bounded simply connected tegion $D \subset R^3$. D is surrounded for $z > 0$ by a characteristic surface $S_3^y: y+x_0 = [x^2+(\int_0^z (k(t))^{1/2}dt)^2]^{1/2}$ and a smooth surface $\sum_4^y: y-x_0 = -[x^2+(\varrho\int_0^z (k(t))^{1/2}dt)^2]^{1/2}$ and for $z < 0$ by a characteristic surface $S_4^x: x-x_0 = -[y^2+(\int_0^z (-k(t))^{1/2}dt)^2]^{1/2}$ and a smooth surface $\sum_3^x: x+x_0 = [y^2+(\varrho\int_0^z (-k(t))^{1/2}dt)^2]^{1/2}$ ($x_0 > 0$; $\varrho \geqslant 1$). All these surfaces intersect the $\{x, y\}$-plane at the straight lines:

$$l_1: x+y+x_0 = 0; \quad l_2: x+y-x_0 = 0; \quad l_3: x-y+x_0 = 0; \quad l_4: x-y-x_0 = 0$$

Under conditions on the boundary $\partial D = (S_4^x \cup \sum_3^x) \cup (\sum_4^y \cup S_3^y)$ and the coefficients $k(z)$, $\lambda(x, y, z)$ we prove the uniqueness of quasi-regular solutions for the boundary value problem $Lu = f$ in D with

$$u|\textstyle\sum_3^x \cup \sum_4^y = 0. \tag{2}$$

The Bi-hyperbolic Degenerate Boundary Value Problem in R^3: consists of finding a solution u of (1) in D assuming prescribed values on $\sum_3^x \cup \sum_4^y$ such that $u \in C^2(\bar{D})$, $k \in C^2(\bar{D})$, $\lambda \in C^1(\bar{D})$, $f \in C^0(\bar{D})$ ([2], [3]).

Definition ([1], pg. 324).

A function $u = u(x, y, z)$ is called *a quasi-regular solution* of (1) if the following conditions hold.

(i) $u(x, y, z)$ satisfies (1)

(ii) The integral $\iint_{D\cap\{z=0\}} [\cdot]\, dx\, dy$ exists

(iii) If $D_i(\varepsilon)$ ($i = 1, 2$) are regions with boundaries $\partial D_i(\varepsilon)$ lying entirely in D_i such that

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$D_1 = D \cap \{z > 0\}$, $D_2 = D \cap \{z < 0\}$, then the boundary integrals along $\partial D_i(\varepsilon)$ which result from the application of Green's theorem to the integrals

$$\iiint_{D_i(\varepsilon)} u_x\, Lu\, dV\,, \qquad \iiint_{D_i(\varepsilon)} u_y\, Lu\, dV \quad (dV = dx\, dy\, dz)$$

have a limit when $\partial D_i(\varepsilon)$ approaches ∂D_i.

Theorem: *Assume the domain $D \subset R^3$ described above. Moreover, we assume that $\lambda_x - \lambda_y \geqslant 0$ in D; and $\lambda \leqslant 0$ on $S_4^x \cup S_3^y$. Then there exists at most one quasi-regular solution of the boundary value problem* (1) *and* (2).

Proof: We employ the following energy integral method. We choose $b \in C^1(\bar{D})$, $c \in C^2(\bar{D})$ appropriate functions of x, y, z, and then we apply Green's theorem to the integral expression:

$$2\,(lu, Lu) = 2.\iiint_D luLu\, dV \tag{3}$$

so that $lu = bu_x + cu_y$ and

$$0 \leqslant \iint_{\partial D} [\cdot]\, dS = \iiint_D [\cdot\cdot]\, dV \leqslant 0 \tag{4}$$

where $[\cdot]$ and $[..]$ are of one sign in ∂D and D, respectively.

We note the following identities:

$$\begin{aligned}
2kbu_x u_{xx} &= (kbu_x^2)_x - kb_x u_x^2\\
2kbu_x u_{yy} &= 2\,(kbu_x u_y)_y - 2kb_y u_x u_y - (kbu_y^2)_x + kb_x u_y^2\\
2bu_x u_{zz} &= 2\,(bu_x u_z)_z - 2b_z u_x u_z - (bu_z^2)_x + b_x u_z^2\\
2kcu_y u_{xx} &= 2\,(kcu_x u_y)_x - 2kc_x u_x u_y - (kcu_x^2)_y + kc_y u_x^2\\
2kcu_y u_{yy} &= (kcu_y^2)_y - kc_y u_y^2\\
2cu_y u_{zz} &= 2\,(cu_y u_z)_z - 2\,(c)_z u_y u_z - (cu_z^2)_y + (c)_y u_z^2\\
2\lambda buu_x &= (b\lambda u^2)_x - (b\lambda)_x u^2\\
2\lambda cuu_y &= (c\lambda u^2)_y - (c\lambda)_y u^2\,.
\end{aligned}$$

(5) B choosing $b = -1$, $c = 1$ in D and then substituting the above identities into (3) and applying Green's theorem we get

$$\begin{aligned}
2\,(lu, Lu) = \iiint_D (\lambda_x - \lambda_y)\, u^2 \cdot dV + \iint_{\sigma D} \lambda\,(v_2 - v_1)\, u^2 dS + \iint_{\sigma D} [(-k)\,(v_1 +\\
+ v_2)\,(u_x^2 + u_y^2) + (v_1 - v_2)\, u_z^2 + 2k\,(v_1 + v_2)\, u_x u_y - 2v_3 u_x u_z + 2v_3 u_y u_z]\, dS =\\
= J_1 + J_2 + J_3\,,
\end{aligned} \tag{6}$$

where $v = (v_1, v_2, v_3)$ is the outer normal vector on D such that

$$\left(k\,(z)\,(v_1^2 - v_2^2) + v_3^2\right)\Big|_{S_3^x \cup S_4^y} = 0\,; \quad \left(k\,(z)\,(v_1^2 - v_2^2) + v_3^2\right)\Big|_{\Sigma_3^x \cup \Sigma_3^y} \geqslant 0\,: \tag{7}$$

$$v_1\big|_{S_4^x} < 0\,;\; v_1\big|_{\Sigma_3^x} < 0\,; \quad v_2\big|_{\Sigma_4^y}\; v_2\big|_{S_3^y} < 0$$

Lemma 1. Suppose (7) is assumed on $S_4^x \cup S_3^y$. Then

$$J_3^{(1)} = \iint_{S_4^x \cup S_3^y} Q(u_x, u_y, u_z)\, dS \geqslant 0, \tag{8}$$

where $Q = Q(u_x, u_y, u_z) = \alpha u_x^2 + \beta u_y^2 + \gamma u_z^2 + 2\delta u_x u_y + 2\varepsilon u_x u_z + 2\zeta u_y u_z$ is a quadratic form with respect to u_x, u_y, u_z such that: $\alpha = \beta = (-k)(v_1 + v_2)$; $\gamma = v_1 - v_2$; $\delta = k(v_1 + v_2)$; $\varepsilon = -v_3$; $\zeta = v_3$.

Proof: To prove that $Q = Q(u_x, u_y, u_z)$ is positive definite we show that all the seven principal minors of the matrix of coefficients of Q are non-negative definite on $S_4^x \cup S_3^y$. These minors are the following expressions:

$\Delta_{11} = \alpha$; $\Delta_{21} = \beta$; $\Delta_{31} = \gamma$

$\Delta_{12} = \beta \cdot \gamma - \zeta^2$; $\Delta_{22} = \alpha \cdot \gamma - \varepsilon^2$; $\Delta_{32} = \alpha \cdot \beta - \delta^2$

$$\Delta_3 = \begin{vmatrix} \alpha & \delta & \varepsilon \\ \delta & \beta & \zeta \\ \varepsilon & \zeta & \gamma \end{vmatrix} = \alpha \cdot \Delta_{12} - \delta(\gamma\delta - \varepsilon\zeta) + \varepsilon(\delta\zeta - \beta\varepsilon).$$

Therefore, $\Delta_{11} = \Delta_{21} = (-k)(v_1 + v_2)$; $\Delta_{31} = v_1 - v_2$;

$$\Delta_{i2} = -kv_i^2(b^2 - c^2) = 0\,; \quad \Delta_3 = -kbv_1(k(v_1^2 - v_2^2) + v_3^2)(b^2 - c^2) = 0 \tag{9}$$

By computing the outer normal vector on $S_4^x \cup S_3^y$ we find:

$$v|_{S_4^x} = (-(x - x_0), y, -N_0^x)\left(N_0^x = -(-k(z))^{1/2} \cdot \int_0^z (-k(t))^{1/2}\, dt\right) \tag{10}$$

$$v|_{S_3^y} = (x, -(y + x_0), N_0^y)\left(N_0^y = (k(z))^{1/2} \cdot \int_0^z (k(t))^{1/2}\, dt\right). \tag{11}$$

From (9) and (10) we get

$$\Delta_{11} = \Delta_{21} = (-k)(-(x - x_0) + y) = (-k)\left(\left[y^2 + \left(\int_0^z (-k(t))^{1/2}\, dt\right)^2\right]^{1/2} + y\right) = \tag{12}$$

$$= (-k)((|y| + \mu_1) + y) \geqslant 0, \text{ where } \mu_1 \geqslant 0. \quad \text{Also: } \Delta_{31} = (|y| + \mu_1) - y \geqslant 0.$$

Similarly from (9) and (11) we find that

$$\Delta_{11} = \Delta_{21} = (-k)(x - (y + x_0)) = (-k)\left(x - \left[x^2 + \left(\int_0^z (k(t))^{1/2}\, dt\right)^2\right]^{1/2}\right) = \tag{13}$$

$$= (-k)(x - (|x| + \mu_2)) \geqslant 0, \text{ where } \mu_2 \geqslant 0. \text{ Also: } \Delta_{31} = x + (|x| + \mu_2) \geqslant 0.$$

Hence, from (9) and (12 – 13) we conclude that $J_3^{(1)} = \iint_{S_4^x \cup S_3^y} y\, Q(u_x, u_y, u_z)\, dS$ is non-negative.

Lemma 2. Suppose (7) is assumed on $\Sigma_3^x \cup \Sigma_4^y$. Then

$$J_3^{(2)} = \iint_{\Sigma_4^x \cup \Sigma_3^y} Q(u_x, u_y, u_z)\, dS \geqslant 0, \tag{14}$$

where $Q = Q(u_x, u_y, u_z)$ is defined as in Lemma 1.

Proof: From (2) it implies that $du = u_x\,dx + u_y\,dy + u_z\,dz = 0$, or

(15) $u_x = Nv_1 :\ u_y = Nv_2\ ;\ u_z = Nv_3$ on $\sum_3^x \cup \sum_4^y$ (N = normalizing factor).

By substituting the expressions (15) into Q we find that

(16) $$Q = Q(u_x, u_y, u_z) = N^2(v_2 - v_1)(k(v_1^2 - v_2^2) + v_3^2) \geqslant 0 \text{ on } \textstyle\sum_3^x \cup \sum_4^y$$

because by computing the outer normal vector v on $\sum_3^x \sum_4^y$ we have

(17) $$v|_{\Sigma_3^x} = \left(-(x+x_0), y, -M_0^x\right)\left(M_0^x = -\varrho^2(-k(z))^{1/2} \cdot \int_0^z (-k(t))^{1/2}\,dt\right)$$

(18) $$v|_{\Sigma_4^y} = \left(x, -(y-x_0), M_0^y\right)\left(M_0^y = \varrho^2(k(z))^{1/2} \cdot \int_0^z (k(t))^{1/2}\,dt\right)$$

and therefore

(19) $$(v_2 - v_1)|_{\Sigma_3^x} = y + (x+x_0) = y + \left[y^2 + \left(\varrho \cdot \int_0^z (-k(t))^{1/2}\,dt\right)^2\right]^{1/2} =$$

and $$= y + (|y| + m_1) \geqslant 0, \text{ where } m_1 \geqslant 0\,.$$

(20) $$(v_2 - v_1)|_{\Sigma_4^y} = -(y-x_0) - x = \left[x^2 + \left(\varrho \cdot \int_0^z (k(t))^{12}\,dt\right)^2\right]^{12} - x =$$

$$= -x + (|x| + m_2) \geqslant 0, \text{ where } m_2 \geqslant 0\,.$$

$J_1 \geqslant 0$ and $J_2 \geqslant 0$ by hypothesis that $\lambda_x - \lambda_y \geqslant 0$ in D and that $\lambda|_{S_4^x \cup S_3^y} \leqslant 0$ and from (2). This fact and Lemmas (1–2) prove our theorem.

(Q.E.D).

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