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ON THE DERIVATIVE OF A POLYNOMIAL

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C. Pommerenke [~~3~~ 2, p. 374] established the following result (suggested by P. Erdös) : If $f(z) = z^n + \ldots$ is a polynomial of degree n in

$$\mathbb{C}^1 = \{z = x + i.y \,/\, (x, y) \in \mathbb{R}^2,\ i = (-1)^{1/2}\}$$

and if the set E on which $|f(z)| < 1$ is connected, then

$$\max_{z \in C} |f'(z)| < e.n^2/2, \tag{1}$$

where $C = \{z : |f(z)| = 1\}$ and

$$\frac{1}{L} \int_C |f'| \, ds \leqslant n$$

(*where* L *is the length of* C), *with equality if and only if* $f(z) = (z + c)^n$. In this paper we prove that

$$\max_{z \in T} |f'(z)| \geqslant n, \tag{2}$$

$$\max_{z \in C} |f'(z)| \geqslant n, \tag{3}$$

where $T = \{z : |z| = 1, z \in \mathbb{C}^1\}$, and C is defined above. To prove inequalities (2) and (3) we apply either the classical Rouché's theorem [3, p. 218, p. 266, p. 279-282], or the maximum modulus principle.

First proof of the (first) inequality (2). We assume $f(z) = z^n + \ldots$, and $f'(z) = n.z^{n-1} + \ldots$. Let us assume that (2) is not true. Therefore,

$$\max_{z \in T} |f'(z)| < n \tag{4}$$

so that (4) yields

$$\begin{aligned} |n.z^{n-1}| &= n.\, |z|^{n-1} = n > \max_{z \in T} |f'(z)| > |f'(z)| \\ &= |[f'(z) - n.z^{n-1}] + n.z^{n-1}|. \end{aligned} \tag{5}$$

Thus from (5) and Rouché's theorem [3, p. 218] we conclude that the polynomials $F = f'(z) - n.z^{n-1}$, and $G = n.z^{n-1}$ have the same number of zeros in the unit disk $U = \{z : |z| < 1, z \in \mathbb{C}^1\}$, $T = \partial U$. Since F is a polynomial of degree at most $n - 2$, this is only possible when $F = 0$, i.e. $f(z) = z^n + c$. This clearly leads to a contradiction. The proof of (2) is therefore complete.

Second proof of the (first) inequality (2). The rational function D defined by $D(z) = f'(z).z^{1-n}$ tends to n as $z \to \infty$. Since $|D(z)| = |f'(z)|$ on the circle T, it follows from the maximum modulus principle that

$$n \leqslant \sup_{z \in T} |D(z)| = \sup_{z \in T} |f'(z)|, \tag{6}$$

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with equality if and only if D is constant outside a compact subset of $\mathbb{C}^1$, i.e. $f(z) = z^n + c$. The proof of (2) is therefore complete.

First proof of the (second) inequality (3). We assume that (3) is not true. Hence,

$$| n^n . [f(z)]^{n-1} | = n^n . | f(z) |^{n-1} = n^n > [\max_{z \in C} | f'(z) |]^n > | f'(z) |^n$$
$$= | \{[f'(z)]^n - n^n . [f(z)]^{n-1}\} + n^n . [f(z)]^{n-1} |.$$

Therefore from Rouché's theorem [3, p. 266] we conclude that the polynomials $F_0 = [f'(z)]^n - n^n . [f(z)]^{n-1}$, and $G_0 = n^n . [f(z)]^{n-1}$ have the same number of zeros in the set $E = \{z : | f(z) | < 1, z \in \mathbb{C}^1\}$, $C = \partial E$. But this leads to a contradiction because the number of zeros of F_0 is at most $n . (n - 1) - 1$. Equality can only occur if $f(z) = (z + c)^n$. The proof of (3) is therefore complete.

Second proof of the (second) inequality (3). We now suppose that E is *connected*. As pointed out by C. Pommerenke [1, p. 222], the function $g(z) = f(z)^{1/n}$ such that $\lim_{z \to \infty} [g(z)/z] = 1$ maps the region $G = \{ | f(z) | > 1\}$ conformally onto $\{ | w | > 1\}$ and can be extended as a homeomorphism of $G \cup C$ onto $\{ | w | \geq 1\}$ [3, p. 279-282]. Let us consider $F(w) = w^{1-n} . f'(h(w))$, where h is the inverse mapping of g. The function F is obviously continuous on $\{ | w | \geq 1\}$, holomorphic in $\{ | w | > 1\}$ and tends to n at ∞, so that, by the maximum modulus principle,

$$n \leq \sup_{|w|=1} | F(w) | = \sup_{z \in C} | f'(z) |.$$

Equality can only occur if F is constant, i.e. $F(w) = n$ for $| w | > 1$. With $w = g(z)$ (with $z \in G$), we get $f'(z) = n . g(z)^{n-1}$ or $n^{-1} . f'(z) . f(z)^{(1-n)/n} = 1$ in G, which implies $f(z) = (z + c)^n$. The converse is obvious. The proof of (3) is therefore complete.

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