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A Maximum Principle in $\mathbb{R}^{n+1}$

JOHN MICHAEL S. RASSIAS

Daphne, Athens, Greece

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In this paper we establish maximum principles of the Cauchy problem for hyperbolic equations in $\mathbb{R}^3$ and $\mathbb{R}^{n+1}$ ($n \geqslant 2$). Our maximum principles generalize the results of Weinberger [5], and Sather [3, 4] for a class of equations such that the coefficients can be allowed to depend upon t, as well, in $\{x_1, x_2, t\}$-space and $\{x_1, x_2, ..., x_n, t\}$-space. Throughout this paper, the influence of the work of Douglis [1] is apparent. See [2].

1. MAXIMUM PRINCIPLE IN $\mathbb{R}^3$

Consider the hyperbolic equation

$$Lu = k(t) \cdot (u_{xx_1} + u_{x_2x_2}) + u_{tt} = f(x, t) \in C^0(\bar{D}), \tag{1}$$

where $x = (x_1, x_2)$ and the domain D is defined as

$$D = \left\{(x, t): T_0 \leqslant t \leqslant T < 0, x_1^2 + x_2^2 < \left(\int_t^T (-k(s))^{1/2}\, ds\right)^2\right\}.$$

Let S_0 be the portion of the boundary of D, which lies in the plane $t = T_0$, and S_4 be the remainder of the boundary G of D, which is a characteristic conoid with respect to (1).

Then let us introduce a solution u of (1) in D which satisfies the initial conditions

$$u(x, T_0) = \tilde{r}(x) \in C^2(\bar{D}), \qquad u_t(x, T_0) = h(x) \in C^0(\bar{D}).$$

With the change of variable

$$v(x, t) = u(x, t) - \tilde{r}(x) \tag{2}$$

Eq. (1) can be written as follows:

$$Lv = k(t) \cdot (v_{x_1x_1} + x_{x_2x_2}) + v_{tt} = F(x, t) \in C^0(\bar{D}), \tag{3}$$

where $F = f - k \cdot (\tilde{r}_{x_1x_1} + \tilde{r}_{x_2x_2})$.

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On the other hand, the initial data above change to

$$v(x, T_0) = 0, \qquad v_t(x, T_0) = h(x) \in C^0(\bar{D}). \tag{4}$$

Assume the transformation to spherical polar coordinates

$$x_1 = r \cdot \cos p, \qquad x_2 = r \cdot \sin p, \tag{5}$$

where

$$p \in [0, 2\pi], \; r = \int_t^T (-k(s))^{1/2} \, ds.$$

By applying (5) we get

$$v_{x_1x_1} + v_{x_2x_2} = v_{rr} + (1/r) \cdot v_r + (1/r^2) \cdot v_{pp}, \tag{6}$$

$$Lv = k(t) \cdot (v_{rr} + (1/r^2) \cdot v_{pp}) + (1/r) \cdot k(t) \cdot v_r + v_{tt} = F_0(r, p, t), \tag{7}$$

where $F_0 \in C^0(\bar{D})$.

Let S_4^e be the part of S_4, where $r_0 > e$, such that

$$r_0 = \int_{T_0}^T (-k(s))^{1/2} \, ds, \qquad e = \int_{T^0}^T (-k(s))^{1/2} \, ds, \tag{8}$$

where $T_0 \leqslant t \leqslant T^0 \leqslant T < 0$.

The direct characteristic conoid S_4 as well as the truncated one S_4^e are generated by the bicharacteristics of space-time (5). The angle parameter p is obviously constant along each fixed generator of S_4.

On the other hand, the total derivative of S_4 is given by

$$d/dr = -(-k)^{-1/2} \cdot (\partial/\partial t - (-k)^{1/2} \cdot \partial/\partial r). \tag{9}$$

We observe that for any smooth function $w = w(r, p, t)$ we have

$$\begin{aligned} w \cdot Lv &= (wv)_{tt} + k \cdot (wv)_{rr} + v_r \cdot k(w/r - 2w_r) - 2w_t v_t \\ &\quad - (kw_{rr} + w_{tt}) \cdot v + (kw/r^2) \cdot v_{pp} \\ &= -(-k)^{1/2} \cdot d(w(v_t + (-k)^{1/2} \cdot v_r)/dr + v(w_t + (-k)^{1/2} \cdot w_r)) \\ &\quad + B(w) \cdot v_r + v \cdot F(w) + D(w, p), \end{aligned} \tag{10}$$

where

$$\begin{aligned} d((wv)_t + (-k)^{1/2} \cdot (wv)_r)/dr &= -(-k)^{-1/2} \cdot (k \cdot (wv)_{rr} + (wv)_{tt} \\ &\quad - (wv)_r \cdot k' \cdot (-k)^{-1/2}) \cdot (1/2)), \end{aligned}$$

$$B(w) = -2k \cdot w_r + (1/r) \cdot w \cdot k + w \cdot k' \cdot (-k)^{-1/2} \cdot (1/2),$$
$$F(w) = -k \cdot w_{rr} - w_{tt} + w_r \cdot k' \cdot (-k)^{-1/2} \cdot (1/2),$$
$$D(w, p) = (kw/r^2) \cdot v_{pp} - 2w_t v_t,$$
$$\tilde{B} = [kw/r^2] \cdot v_{pp}, \qquad \tilde{D} = -2w_t \cdot v_t.$$

We set as

$$w = w(r, p, t) = r^{1/2} \cdot (-k)^{-1/4}, \tag{11}$$

such that $-2k \cdot w_r + (1/r) \cdot kw$ vanishes.

Then the identity (10) becomes, on S_4^e,

$$w \cdot Lv = -(-k)^{1/2} \cdot d(d^+(wv))/dr + G(w) + v \cdot F(w), \tag{12}$$

where $d^+ = \partial/\partial t + (-k)^{1/2} \cdot \partial/\partial r$ such that $d^+(wv) = w \cdot d^+(v) + v \cdot d^+(w)$, and $G(w) = B(w) \cdot v_r + \tilde{D}(w, v)$.

Assume the operators

$$R_e v = \int_e^{r_0} dr \cdot \int_0^{2\pi} v(r, p, t)\, dp \qquad \text{on } S_4^e. \tag{13}$$

In addition, we assume the following condition

$$V(p) = \int_{p=0}^{2\pi} v_{pp}\, dp = 0 \qquad \text{on } S_4^e. \tag{+}$$

Then we have the following properties:

(i) $R_e\tilde{B} = 0$ because of (+). Define

$$Rv = \lim_{e \to 0} R_e v \tag{14}$$

when the limit exists. Then $R\tilde{B} = 0$.

(ii) If $Q = (r_0, p, T_0)$ and $Q_e = (e, p, T^0)$, then for $P = P(0, 0, T(<0))$

$$R_e[(-k)^{1/2} \cdot d(d^+(wv))/dr] = \int_0^{2\pi} dp \int_e^{r_0} v(P) \cdot d(d^+(w))/dr \cdot dr$$
$$\cdot (-k(t))^{1/2} + (-k(t))^{1/2} \cdot \int_0^{2\pi} w\, d^+(v) \Big|_{Q_e}^{Q} dp$$
$$+ (-k)^{1/2} \cdot \int_0^{2\pi} (v - v(P)) \cdot d^+(w) \Big|_{Q_e}^{Q} dp, \tag{15}$$

$$\lim_{e\to 0}\left(\int_0^{2\pi} w\cdot d^+(v)\Big|_{Q_e} dp\right)=\lim_{e\to 0}\left(\int_0^{2\pi}(v-v(P))\cdot d^+(w)\Big|_{Q_e} dp\right)=0, \tag{16}$$

$$\begin{aligned} R[(-k)^{1/2}\cdot d(d^+(wv))/dr] &= \int_0^{2\pi} dp\cdot\int_0^{r_0} v(P)\cdot d(d^+(w))/dr\cdot dr \\ &\quad\cdot(-k)^{1/2}+(-k)^{1/2}\cdot\int_0^{2\pi} w\cdot d^+(v)\Big|_Q dp \\ &\quad+(-k)^{1/2}\cdot\int_0^{2\pi}(v-v(P))\cdot d^+(w)\Big|_Q dp \end{aligned} \tag{17}$$

(iii) Similarly, we find

$$\begin{aligned} R(G(w)) &= -\int_0^{2\pi} dp\cdot\int_0^{r_0}(v-v(P))\cdot d(wk'/2(-k)^{1/2})/dr\cdot dr \\ &\quad+\int_0^{2\pi}(wk'/2(-k)^{1/2})\cdot(v-v(P))\Big|_Q dp. \end{aligned} \tag{18}$$

(iv) Finally,

$$R(v\cdot F(w))=\int_0^{2\pi} dp\cdot\int_0^{r_0} v\cdot d(d^+(w))/dr\cdot dr\cdot(-k)^{1/2}. \tag{19}$$

Substituting (i)–(iv) into (12) we get the following fundamental integral identity

$$\begin{aligned} R(w\cdot Lv) &= \int_0^{2\pi} dp\cdot\int_0^{r_0}(v-v(P))\cdot[d(d^+(w))/dr\cdot(-k)^{1/2} \\ &\quad-d(w\cdot k'/2(-k)^{1/2})/dr]\,dr \\ &\quad-\int_0^{2\pi} w[(-k)^{1/2}\cdot d^+(v)-(v-v(P))\cdot k'/2(-k)^{1/2}]\Big|_Q dp \\ &\quad-(-k)^{1/2}\cdot\int_0^{2\pi}(v-v(P))\cdot d^+(w)\Big|_Q dp. \end{aligned} \tag{20}$$

THEOREM 1. *Assume that $v(r,p,t)\in C^2(\bar{D})$ satisfies the differential inequality $Lv=F_0\leqslant 0$, and that $v(r,p,T_0)=0$, $v_t(r,p,T_0)=h(r,p)\leqslant 0$, where D is defined above, and assume the following condition*:

$$d(w\cdot k'/2(-k)^{1/2})/dr-d(d^+(w))/dr\cdot(-k)^{1/2}\geqslant 0, \tag{21}$$

and (+). Then $v(r,p,t)\leqslant 0$ in D.

Proof. Let P be the point with the smallest t-coordinate T for which v vanishes. The identity (20) implies

$$0 \geqslant -\int_0^{2\pi} w \cdot v_t \bigg|_Q dp, \tag{22}$$

which is a contradiction. By continuity v cannot become positive even when $v_t \leqslant 0$ is not strict. Therefore, under the condition (+) and condition (21) the operator L has a maximum property in D. Q.E.D.

THEOREM 2. *Assume that $v(r, p, t) \in C^2(\bar{D})$ satisfies the differential inequality*

$$\begin{aligned} Lv = k(t) \cdot (v_{rr} + (1/r^2) \cdot v_{pp}) + (1/r) \cdot k(t) \cdot v_r + v_{tt} + a(r, p, t) \cdot v_r \\ + b(r, p, t) \cdot v + c(r, p, t) \cdot v_t + m(r, p, t) \cdot v = F_0^0(r, p, t), \end{aligned} \tag{23}$$

where a, b, c, and m are continuously differentiable functions of the independent variables r, p, t in D described above, and that $v(r, p, t) = 0$, $v_t(r, p, t) \leqslant 0$, $F_0^0 \in C^0(\bar{D})$. Assume, in addition, the conditions

$$\begin{aligned} d\left[w \cdot \left(k'/2(-k)^{1/2} + \left(a - \int_0^r a_r \cdot dr\right)\right)\right]\bigg/dr \\ - d(d^+(w))/dr \cdot (-k)^{1/2} \geqslant 0, \end{aligned} \tag{24}$$

$$c = k^{-1} \cdot \int_0^r a\, dr - (-k)^{-1/2} \cdot \left(a - \int_0^r a_r\, dr\right), \tag{25}$$

$$b_p - m + \int_0^r (a)_p/2k\, dr \geqslant 0, \tag{26}$$

$$w = r^{1/2} \cdot (-k)^{1/4} \cdot \exp\left(\int_0^r a/2k \cdot dr\right), \tag{27}$$

and (+). *Then $v(r, p, t) \leqslant 0$ in D.*

Proof. We prove in the same way as in the case of (20) that the following fundamental integral identity holds, as well:

$$\begin{aligned} R(w \cdot Lv) = \int_0^{2\pi} dp \cdot \int_0^{r_0} (v - v(P)) \cdot \bigg[d(d^+(w))/dr \cdot (-k)^{1/2} \\ - d\left(wk'/2(-k)^{1/2} + \left(a - \int_0^r a_r \cdot dr\right)\right)\bigg/dr\bigg]\, dr \end{aligned}$$

$$+\int_0^{2\pi} dp \cdot \int_0^{r_0} v \cdot (m \cdot w - (w \cdot b)_p)\, dr - \int_0^{2\pi} w \left[d^+(v) \cdot (-k)^{1/2} \right.$$

$$\left. - \left(k'/2(-k)^{1/2} + \left(a - \int_0^r a_r\, dr\right)\right) \cdot (v - v(P))\right]\Bigg|_Q dp$$

$$- (-k)^{1/2} \cdot \int_0^{2\pi} (v - v(P)) \cdot d^+(w)\Bigg|_Q dp. \tag{28}$$

The proof now is obvious. Q.E.D.

Remark. To have $v_t < 0$ given $v_t \leqslant 0$, we choose

$$v_0 = v(r, p, t) - c_1 \cdot \exp(c_2 \cdot t) < 0, \qquad \text{for} \quad c_i, i = 1, 2, \text{ both positive},$$

such that $c_2^2 + c_2 \cdot c + m > 0$.

2. Maximum Principle in $\mathbb{R}^{n+1}$ ($n \geqslant 2$)

Consider the hyperbolic equation

$$Lu = k(t) \cdot \sum_{i=1}^{n} u_{x_i x_i} + u_{tt} = f(x, t) \in C^0(\bar{D}), \tag{29}$$

where $x = (x_1, x_2, ..., x_n)$ and the domain D is defined as

$$D = \left\{(x, t)\colon T_0 \leqslant t \leqslant T < 0, \sum_{i=1}^{n} x_i^2 < \left(\int_t^T (-k(s))^{1/2}\, ds\right)^2\right\}.$$

Let S_0 be the portion of the boundary of D, which lies in the plane $t = T_0$, and S_4 be the remainder of the boundary G of G, which is a characteristic conoid with respect to (29).

THEOREM 3. *Assume that $v(r, p, q, t) \in C^2(\bar{D})$ satisfies the differential inequality*

$$Lv = k(t) \cdot \left(v_{rr} + g^{-1}\left(\partial(g_0 v_p/r^2)/\partial p + \sum_{i=1}^{n-2} \partial(g_i v_{q_i})/\partial q_i\right)\right)$$

$$+ [(n-1)/r] \cdot k(t) \cdot v_r + v_{tt} + a_1(r, p, q, t) \cdot v_r$$

$$+ a_2(r, p, q, t) \cdot v + \sum_{i=1}^{n-2} a_{i+2}(r, p, q, t) \cdot v_{q_i}$$

$$+ b(r, p, q, t) \cdot v_t + m(r, p, q, t) \cdot v = F(r, p, q, t) \leqslant 0, \tag{30}$$

where a_i, $i = 1, 2, \ldots, n$, b, and m are continuously differentiable functions of the independent variables r, p, q, t in D described above, and that $v(r, p, q, t) = 0$, $v_t(r, p, q, t) \leqslant 0$, $F \in C^0(\bar{D})$, and

$$\begin{aligned}
x_1 &= r \cdot \cos p \cdot \sin q_1 \cdot \sin q_2 \cdots \sin q_{n-2},\\
x_2 &= r \cdot \sin p \cdot \sin q_1 \cdot \sin q_2 \cdots \sin q_{n-2},\\
&\vdots\\
x_{n-1} &= r \cdot \cos q_{n-3} \cdot \sin q_{n-2},\\
x_n &= r \cdot \cos q_{n-2}, \qquad p \in [0, 2\pi],\\
q &= (q_1, q_2, \ldots, q_{n-2}), \qquad q_i \in [0, \pi], \quad i = 1, 2, \ldots, n-2,\\
g &= \sin q_1 \cdot \sin^2 q_2 \cdots \sin^{n-2} q_{n-2} = g_{n-2},\\
g_0 &= \operatorname{cosec} q_1 \cdot \sin q_3 \cdot \sin^2 q_4 \cdots \sin^{n-4} q_{n-2},\\
g_1 &= \sin q_1 \cdot \sin q_3 \cdot \sin^2 q_4 \cdots \sin^{n-4} q_{n-2},\\
g_2 &= \sin q_1 \cdot \sin^2 q_2 \cdot \sin q_3 \cdot \sin^2 q_4 \cdots \sin^{n-4} q_{n-2},\\
g_3 &= \sin q_1 \cdot \sin^2 q_2 \cdot \sin^3 q_3 \cdot \sin^2 q_4 \cdot \sin^3 q_5 \cdots \sin^{n-4} q_{n-2},\\
g_{n-3} &= \sin q_1 \cdot \sin^2 q_2 \cdots \sin^{n-3} q_{n-3} \cdot \sin^{n-4} q_{n-2},\\
g_{n-2} &= g.
\end{aligned}$$

Assume, in addition the conditions

$$d\left[w \cdot \left(k'/2(-k)^{1/2} + \left(a_1 - \int_0^r (a_1)_r\, dr\right) + (-k)^{1/2} \cdot \left(b - \int_0^r b_r\, dr\right)\right)\right]\Big/ dr - (-k)^{1/2} \cdot d(d^+(w) + b \cdot w)/dr \geqslant 0, \tag{31}$$

$$[k'/2(-k)^{1/2}] + \left(a_1 - \int_0^r (a_1)_r\, dr\right) + (-k)^{1/2} \cdot \left(b - \int_0^r b_r\, dr\right) = (-k)^{1/2} \cdot \partial\left(\int_0^r (a_1 + (-k)^{1/2} \cdot b)/k \cdot dr\right]\Big/\partial t - k'/2k, \tag{32}$$

$$\sum_{i=1}^{n-2} \int_0^r ((a_1)_{q_i} + (-k)^{1/2} \cdot b_{q_i})/2k\, dr + (a_2)_p - m + \sum_{i=1}^{n-2} (a_{i+2})_{q_i} + \int_0^r ((a_1)_p + (-k)^{1/2} \cdot b_p)/2k\, dr \geqslant 0, \tag{33}$$

$$w = r^{(n-1)/2} \cdot (-k)^{-1/4} \cdot \exp\left(\int_0^r (a_1 + (-k)^{1/2} \cdot b)/2k\, dr\right) \cdot \sin q_1 \cdot \sin^2 q_2 \cdots \sin^{n-2} q_{n-2}, \tag{34}$$

$$d^+ = \partial/\partial t + (-k)^{1/2} \cdot \partial/\partial r, \tag{35}$$

and

$$\int_{p=0}^{2\pi} v_{pp}\, dp = \int_{q_i=0}^{\pi} v_{q_i q_i}\, dq_i = 0, \qquad i = 1, 2, \ldots, n-2, \text{ on } S_4^e, \tag{36}$$

where S_4^e is the part of S_4, such that $r_0 > e$. Then $v(r, p, q, t) \leqslant 0$ in D.

Proof. We define w as in (34) in order that the expression $-2k \cdot w_r + (n-1) \cdot k \cdot w/r$ vanishes if $a_1 = b = 0$. Let us assume the operators

$$R_e v = \int_e^{r_0} dr \cdot \oint v(r, p, q, t)\, dS \qquad \text{on } S_4^e, \tag{37}$$

$$dS = g \cdot dq_1 \cdot dq_2 \cdots dq_{n-2}, \tag{38}$$

and

$$Rv = \lim_{e \to 0} R_e v, \tag{39}$$

when the limit exists. Then the proof follows. Q.E.D.

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