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## Special Issue on **Leonhard Paul Euler**'s: Mixed Type Partial Differential Equations (MT. PDE)

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# Special Issue on **Leonhard Paul Euler**'s: Functional Equations and Inequalities (F. E. I.)

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# International Journal of Applied Mathematics & Statistics


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# PREFACE

This Euler's commemorating volume entitled :
***Functional Equations , Integral Equations, Differential Equations and Applications*** (**F. I. D. A**), is a forum for exchanging ideas among eminent mathematicians and physicists, from many parts of the world, as a tribute to the tri-centennial birthday anniversary of Leonhard Paul Euler (April 15,1707 A.D., b. in Basel – September 18, 1783 A.D., d. in St. Petersburg).
*This 998 pages long collection* is composed of outstanding contributions in mathematical and physical equations and inequalities and other fields of mathematical, physical and life sciences.
In addition, this anniversary volume is unique in its target, as it strives to represent a broad and highly selected participation from across and beyond the scientific and technological country regions. It is intended to boost the cooperation among mathematicians and physicists working on a broad variety of pure and applied mathematical areas.
Moreover, this new volume will provide readers and especially researchers with a detailed overview of many significant insights through advanced developments on Euler's mathematics and physics. This transatlantic collection of mathematical ideas and methods comprises a wide area of applications in which equations, inequalities and computational techniques pertinent to their solutions play a core role.
Euler's influence has been tremendous on our everyday life, because new tools have been developed, and revolutionary research results have been achieved , bringing scientists of exact sciences even closer, by fostering the emergence of new approaches, techniques and perspectives.
The central scope of this commemorating 300birthday anniversary volume is broad, by deeper looking at the impact and the ultimate role of mathematical and physical challenges, both inside and outside research institutes, scientific foundations and organizations.
We have recently observed a more rapid development in the areas of research of Euler worldwide. Leonhard P. Euler (1707-1783) was actually the most influential mathematician and prolific writer of the eighteenth century, by having contributed to almost all the fundamental fields of mathematics and mathematical physics. In calculus of variations, according to C. Caratheodory, Euler's work: *Methodus inveniendi lineas curves…*(1740 A.D.) was one of the most beautiful works ever written. Euler was dubbed *Analysis Incarnate* by his peers for his incredible ability. He was especially great from his writings and that produced more academic work on mathematics than anyone. He could produce an entire new mathematical paper in about thirty minutes and had huge piles of his works lying on his desk. It was not uncommon to find Analysis Incarnate ruminating over a new subject with a child on his lap.
This volume is suitable for graduate students and researchers interested in functional equations, integral equations and differential equations and would make an ideal supplementary reading or independent study research text.
*This item will also be of interest to those working in other areas of mathematics and physics. It is a work of great interest and enjoyable read as well as unique in market.*

This Euler's volume (F. I. D. A.) consists of six (6) issues containing various parts of contemporary pure and applied mathematics with emphasis to Euler's mathematics and physics.
It contains sixty eight (68) fundamental research papers of one hundred one (101) outstanding research contributors from twenty seven (27) different countries.
In particular, these contributors come from:
Algerie (1 contributor); Belgique (2); Bosnia and Herzegovina (2); Brazil (2); Bulgaria (3); China (9); Egypt (1); France (3); Greece (2); India (8); Iran (3); Italy (1); Japan (7); Korea (7); Morocco (3); Oman (2); Poland (3); R. O. Belarus (8); Romania (2); Russia (3); Saudi Arabia (1); Serbia and Montenegro (5); The Netherlands (3); U. A. Emirates (1);U. K. (2); U. S. A. (15); Uzbekistan (2).

First Issue (F. E. I.) consisting of 14 research papers, 181 pages long, contains various parts of ***Functional Equations and Inequalities*****,**
namely:
Euler's Life and Work, Ulam stability, Hyers – Ulam stability and Ulam – Gavruta - Rassias stability of functional equations, Euler – Lagrange type and Euler – Lagrange – Rassias quadratic mappings in Banach and Hilbert spaces, Aleksandrov and isometry Ulam stability problems, stability of Pexider and Drygas functional equations, alternative of fixed point, and Hyers - Ulam stability of differential equations.

Second Issue (MT. PDE) consisting of 9 research papers, 117 pages long, contains various parts of ***Mixed Type Partial Differential Equations*,**
namely:
Tricomi - Protter problem of nD mixed type partial differential equations, solutions of generalized Rassias' equation, degenerated elliptic equations, mixed type oblique derivative problem, Cauchy problem for Euler – Poisson - Darboux equation, non - local boundary value problems, non-uniqueness of transonic flow past a flattened airfoil, multiplier methods for mixed type equations.
Third Issue (F . D . E.) consisting of 9 research papers, 146 pages long, contains various parts of ***Functional and Differential Equations*,**
namely:
Iterative method for singular Sturm - Liouville problems, Euler type boundary value problems in quantum mechanics, positive solutions of boundary value problems, controllability of impulsive functional semi-linear differential inclusions in Frechet spaces, asymptotic properties of solutions of the Emden-Fowler equation, comparison theorems for perturbed half-linear Euler differential equations, almost sure asymptotic estimations for solutions of stochastic differential delay equations, difference equations inspired by Euler's discretization method, extended oligopoly models.
Fourth Issue (D. E. I.) consisting of 9 research papers, 160 pages long, contains various parts of ***Differential Equations and Inequalities*,**
namely:
New spaces with wavelets and multi-fractal analysis, mathematical modeling of flow control and wind forces, free convection in conducting fluids, distributions in spaces, strong stability of operator–differential equations, slope – bounding procedure, sinc methods and PDE, Fourier type analysis and quantum mechanics.
Fifth Issue (DS. IDE.) consisting of 9 research papers, 159 pages long, contains various parts of ***Dynamical Systems and Integro - Differential Equations*,**
namely:
Semi-global analysis of dynamical systems, nonlinear functional-differential and integral equations, optimal control of dynamical systems, analytical and numerical solutions of singular integral equations, chaos control of classes of complex dynamical systems, second order integro-differential equation, integro-differential equations with variational derivatives generated by random partial integral equations, inequalities for positive operators, strong convergence for a family of non-expansive mappings.
Sixth Issue (M. T. A.) consisting of 18 research papers, 231 pages long, contains various parts of ***Mathematical Topics and Applications*,**
namely:
Maximal subgroups and theta pairs in a group, Euler constants on algebraic number fields, characterization of modulated Cox measures on topological spaces, hyper-surfaces with flat r-mean curvature and Ribaucour transformations, Leonhard Euler's methods and ideas live on in the thermodynamic hierarchical theory of biological evolution, zeroes of L-series in characteristic p,
Beck's graphs, best co-positive approximation function, Convexity in the theory of the Gamma function, analytical and differential – algebraic properties of Gamma function, Ramanujan's summation formula and related identities, ill – posed problems, zeros of the q-analogues of Euler polynomials, Eulerian and other integral representations for some families of hyper-geometric polynomials, group C*-algebras and their stable rank, complementaries of Greek means to Gini means, class of three- parameter weighted means, research for Bernoulli's inequality.

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**John Michael Rassias**

*Special Editor-in-Chief* of Euler's volume F. I. D. A. – **IJAMAS**.
National and Capodistrian University of Athens, Greece
E-mail: jrassias@primedu.uoa.gr; jrassias@tellas.gr
URL: http://www.primedu.uoa.gr/~jrassias/

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# Initial-Oblique Derivative Problem for Nonlinear Parabolic Equations in High Dimensional Domains

**Dechang Chen**[1] **and Guochun Wen**[2]

[1]Department of Preventive Medicine and Biometrics
Uniformed Services University of the Health Sciences
MD 20814, USA
E-mail: dchen@usuhs.mil

[2]School of Mathematical Sciences
Peking University
Beijing 100871, China
E-mail: wengc@math.pku.edu.cn

**ABSTRACT**

*In this paper, we discuss some initial-boundary value problems for nonlinear nondivergent parabolic equations of second order in high dimensional domains, where coefficients of equations are measurable in multiply connected domains. We focus on initial-oblique derivative problems. The estimates of solutions for the initial-boundary value problems are given, and then the solvability result is derived. The results in this paper are the development of the corresponding work in [1, 3, 4].*

**Keywords:** initial-oblique derivative problems, nonlinear parabolic equations, high dimensional domains.

**2000 Mathematics Subject Classification:** 35K15, 35K20, 35K60.

## 1 Formulation of Initial-Boundary Value Problems for Nonlinear Parabolic Equations

Let $\Omega$ be a bounded multiply connected domain in $\mathbf{R}^N$ with the boundary $\partial\Omega \in C^2_\alpha (0 < \alpha < 1)$. Set $Q = \Omega \times I$, where $I$ is the interval $0 < t \le T, 0 < T < \infty$. Let $\partial Q = \partial Q_1 \cup \partial Q_2$ or $S = S_1 \cup S_2$ denote the parabolic boundary, where $\partial Q_1 = S_1 = \Omega \times \{t = 0\}$ is the bottom and $\partial Q_2 = S_2 = \partial\Omega \times \bar{I}$ is the lateral boundary. We consider the nonlinear parabolic equation of second order

$$F(x, t, u, D_x u, D_x^2 u) - Hu_t = 0 \ \text{ in } Q,$$

where $H$ is a positive number, and $D_x u = (u_{x_i}), D_x^2 u = (u_{x_i x_j})$. Under certain conditions, the above equation can be written as

$$\sum_{i,j=1}^{N} a_{ij} u_{x_i x_j} + \sum_{i=1}^{N} b_i u_{x_i} + cu - Hu_t = f \ \text{ in } Q, \tag{1.1}$$

where

$$a_{ij} = \int_0^1 F_{\tau r_{ij}}(x,t,u,p,\tau r)\mathrm{d}\tau, \quad b_i = \int_0^1 F_{\tau p_i}(x,t,u,\tau p,0)\mathrm{d}\tau,$$

$$c = \int_0^1 F_{\tau u}(x,t,\tau u,0,0)\mathrm{d}\tau, \quad f = -F(x,t,0,0,0),$$

$$\text{with } r = D_x^2 u = (r_{ij}) = \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big), \quad p = D_x u = (p_i) = \Big(\frac{\partial u}{\partial x_i}\Big).$$

Suppose that (1.1) satisfies Condition $C$, i.e. for arbitrary functions $u_1(x,t), u_2(x,t) \in C^{1,0}_{\beta,\beta/2}(\overline{Q}) \cap \tilde{W}_2^{2,1}(Q)$ $(0<\beta<1)$, $F(x,t,u,D_x u, D_x^2 u)$ satisfies the condition

$$F(x,t,u_1,D_x u_1, D_x^2 u_1) - F(x,t,u_2,D_x u_2,D_x^2 u_2)$$
$$= \sum_{i,j=1}^N \tilde{a}_{ij} u_{x_i x_j} + \sum_{i=1}^N \tilde{b}_i u_{x_i} + \tilde{c} u,$$

where $u = u_1 - u_2$, $\tilde{W}_2^{2,1}(Q) = W_2^{2,0}(Q) \cap W_2^{0,1}(Q)$, and

$$\tilde{a}_{ij} = \int_0^1 F_{u_{x_i x_j}}(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau, \quad \tilde{b}_i = \int_0^1 F_{u_{x_i}}(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau,$$

$$\tilde{c} = \int_0^1 F_u(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau, \quad \tilde{u} = u_2 + \tau(u_1 - u_2),$$

$$\tilde{p} = D_x[u_2 + \tau(u_1-u_2)], \quad \tilde{r} = D_x^2[u_2 + \tau(u_1 - u_2)].$$

We assume that $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}, f$ are measurable in $Q$ and satisfy the conditions

$$q_0 \sum_{j=1}^N |\xi_j|^2 \le \sum_{i,j=1}^N \tilde{a}_{ij}\xi_i\xi_j \le q_0^{-1} \sum_{j=1}^N |\xi_j|^2, \quad 0 < q_0 < 1, \tag{1.2}$$

$$\sup_Q \sum_{i,j=1}^N \tilde{a}_{ij}^2(x,t) / \inf_Q \Big[\sum_{i=1}^N \tilde{a}_{ii}(x,t)\Big]^2 \le q_1 < \frac{1}{N-1/2}. \tag{1.3}$$

$$|\tilde{a}_{ij}| \le k_0, \quad |\tilde{b}_i| \le k_0, \quad i,j=1,...,N, \quad |\tilde{c}| \le k_0 \text{ in } Q, \quad L_p[f,\overline{Q}] \le k_1, \tag{1.4}$$

in which $k_0, k_1, q_0, q_1, p\,(> N+2)$ are non-negative constants.

The condition (1.3) may be explained as follows. Consider the linear case of parabolic equation (1.1), namely

$$\sum_{i,j=1}^N a_{ij}(x,t) u_{x_i x_j} + \sum_{i=1}^N b_i(x,t) u_{x_i} + c(x,t) u - H u_t = f(x,t) \text{ in } Q. \tag{1.5}$$

Divide the above equation by $\Lambda = \tau \inf_Q \sum_{i=1}^N a_{ii}$, where $\tau$ is an undetermined positive constant. Denote $\hat{a}_{ij} = a_{ij}/\Lambda$, $\hat{b}_i = b_i/\Lambda\,(i,j=1,\cdots,N)$, $\hat{c} = c/\Lambda$, $\hat{f} = f/\Lambda$. Then the above equation is reduced to the form

$$\hat{\mathcal{L}} u = \sum_{i,j=1}^N \hat{a}_{ij}(x,t) u_{x_i x_j} + \sum_{i=1}^N \hat{b}_i(x,t) u_{x_i} + \hat{c}(x,t) u - H u_{t\Lambda} = \hat{f}, \text{ i.e.}$$

$$Lu = \Delta u - H u_{t\Lambda} = -\sum_{i,j=1}^N [\hat{a}_{ij}(x,t) - \delta_{ij}] u_{x_i x_j} - \sum_{i=1}^N \hat{b}_i(x,t) u_{x_i} - \hat{c}(x,t) u - \hat{f} \text{ in } Q.$$

We require that the above coefficients satisfy

$$\begin{aligned}&\sup_Q[2\sum_{i,j=1,i<j}^N \hat{a}_{ij}^2+\sum_{i=1}^N(\hat{a}_{ii}-1)^2]=\sup_Q[\sum_{i,j=1}^N \hat{a}_{ij}^2+N-2\sum_{i=1}^N\hat{a}_{ii}]<\frac{1}{2},\ \text{i.e.}\\ &\sup_Q[\sum_{i,j=1}^N \hat{a}_{ij}^2-2\sum_{i=1}^N \hat{a}_{ii}]<\frac{1}{2}-N,\end{aligned} \tag{1.6}$$

with the constant $\tau=2/(2N-1)$, and hence we can give the condition (1.3). In fact, consider

$$\sup_Q\sum_{i,j=1}^N \hat{a}_{ij}^2-2\inf_Q\sum_{i=1}^N \hat{a}_{ii}<\frac{1}{2}-N,\ \ \text{i.e.}$$
$$\frac{\sup_Q\sum_{i,j=1}^N a_{ij}^2}{\tau^2\inf_Q[\sum_{i=1}^N a_{ii}]^2}<\frac{2}{\tau}+\frac{1}{2}-N,\ \text{ or }\ \frac{\sup_Q\sum_{i,j=1}^N a_{ij}^2}{\inf_Q[\sum_{i,j=1}^N a_{ii}]^2}<f(\tau)$$

for $f(\tau)=2\tau+(1/2-N)\tau^2$. It is seen that the maximum of $f(\tau)$ on $[0,\infty)$ occurs at the point $\tau=2/(2N-1)$, and the maximum equals $f(2/(2N-1))=1/(N-1/2)$. The above inequality with $\tau=2/(2N-1)$ is just the inequality (1.3). From the inequality it follows that (1.6) with $\tau=2/(2N-1)$ holds.

In this paper we mainly consider the nonlinear parabolic equations of second order

$$\sum_{i,j=1}^N a_{ij}u_{x_ix_j}+\sum_{i=1}^N b_iu_{x_i}+\tilde{c}u-Hu_t=f\ \text{ in }\ Q, \tag{1.7}$$

where $\tilde{c}(x,t)=c(x,t)-|u|^{\sigma_0}$ for any positive constant $\sigma_0$. If equation (1.1) satisfies Condition $C$, then equation (1.7) will be said to satisfy Condition $C'$.

The so-called initial-mixed boundary value problem (Problem $M$) is to find a continuously differentiable solution $u=u(x,t)\in B^*=C^{1,0}_{\beta,\beta/2}(Q)\cap\tilde{W}^{2,1}_2(Q)$ satisfying the initial-boundary conditions

$$u(x,0)=g(x),\ \ x\in\Omega, \tag{1.8}$$

$$\begin{aligned}&lu=d\,\frac{\partial u}{\partial\nu}+\sigma u=\tau(x,t),\ \ (x,t)\in S_2,\ \ \text{i.e.}\\ &lu=\sum_{j=1}^N d_j\frac{\partial u}{\partial x_j}+\sigma u=\tau(x,t),\ \ (x,t)\in S_2,\end{aligned} \tag{1.9}$$

in which $g(x),d(x,t),d_j(x,t)(j=1,...,N),\sigma(x,t),\tau(x,t)$ satisfy the conditions

$$\begin{aligned}&C^2_\alpha[g(x),\bar{\Omega}]\le k_2,\ \ C^{1,1}_{\alpha,\alpha/2}[\sigma(x,t),S_2]\le k_0,\\ &C^{1,1}_{\alpha,\alpha/2}[d_j(x,t),S_2]\le k_0,\ \ C^{1,1}_{\alpha,\alpha/2}[\tau(x,t),S_2]\le k_2,\\ &\cos(\nu,\mathbf{n})\ge q_0>0,\ \ d\ge 0,\ \ \sigma\ge 0,\ \ d+\sigma\ge 1,\ \ (x,t)\in S_2,\end{aligned} \tag{1.10}$$

where $\mathbf{n}$ is the unit outward normal on $S_2,\alpha,\beta\,(0<\beta\le\alpha<1),k_0,k_2,\,q_0(0<q_0<1)$ are non-negative constants. In particular, if Problem $M$ meets the conditions $d=0,\nu=\mathbf{s},\ \sigma=1$ on $S_2$, then Problem $M$ is the Dirichlet boundary value problem, which will be called Problem $D$. If

Problem $M$ meets the conditions $d > 0$ on $S_2$, then Problem $M$ is the initial-regular derivative boundary value problem, which will be called Problem $O$. Problem $O$ with the condition $\nu = \mathbf{n}$, $\sigma = 0$ on $S_2$ is called Problem $N$. In this paper, we mainly discuss Problem $O$ for equation (1.7).

**Theorem 1.1.** *If equation* $(1.7)$ *with* $d = 0$ *satisfies Condition* $C'$*, then Problem O for the equation only has the trivial solution.*

*Proof.* Assume that (1.7) satisfies Condition $C'$. Let $u(x,t)$ be a solution of Problem $O$ for (1.7). It is easy to see that $u(x,t)$ satisfies the equation and the boundary conditions:

$$\sum_{i,j=1}^{N} a_{ij} u_{x_i x_j} + \sum_{i=1}^{N} b_i u_{x_i} + \tilde{c} u - H u_t = 0 \text{ in } Q, \tag{1.11}$$

$$u(x,0) = 0 \text{ on } \Omega, \tag{1.12}$$

$$lu(x,t) = 0, \text{ i.e. } d\frac{\partial u}{\partial \nu} + \sigma u = 0 \text{ on } S_2, \tag{1.13}$$

where $\tilde{c} = c - |u|^{\sigma_0}$. Introduce a transformation $v = u \exp(-Bt)$, where $B$ is an appropriately large number such that $B > \sup_Q \tilde{c}$. Then the initial-boundary value problem (1.11)–(1.13) is reduced to

$$\sum_{i,j=1}^{n} a_{ij} v_{x_i x_j} + \sum_{i=1}^{n} b_i v_{x_i} - [B - \tilde{c}] v - H v_t = 0 \text{ in } Q, \tag{1.14}$$

$$v(x,0) = 0 \text{ on } \Omega, \tag{1.15}$$

$$lv(x,t) = 0, \text{ i.e. } d\frac{\partial v}{\partial \nu} + \sigma v = 0 \text{ on } S_2. \tag{1.16}$$

Noting that $B - \sup_Q \tilde{c} > 0$, $(x,t) \in Q$, there is no harm assuming that $\sigma(x,t) > 0$ on $S_2 \cap \{(x,t) \in S_2, d \neq 0\}$. Otherwise, through a transformation $V(x,t) = v(x,t)/\Psi(x,t)$, where $\Psi(z,t)$ is a solution of the equation

$$\Delta v - H v_t = 0 \text{ in } D, \text{ i.e. } \sum_{j=1}^{n} v_{x_j^2} - H v_t = 0 \text{ in } D$$

with the boundary condition $\Psi(z,t) = 1$ on $\partial Q$, the requirement can be realized and the modified equation satisfies the condition similar to Condition $C$. By the extremum principle of solutions for (1.14) (see Theorems 2.5 and 2.7, Chapter I, [4]), we can derive that $v(x,t) = u(x,t) = 0$. □

In Sections 2 and 3, we shall give a priori estimates of solutions for Problem $O$ and prove, by using the Leray-Schauder theorem, the existence of solutions for Problem $O$.

## 2 A Priori Estimates of Solutions for Initial-Oblique Derivative Problems

In the following, we give the estimates of $C^{1,0}(\bar{Q})$ and $C^{1,0}_{\beta,\beta/2}(\overline{Q})$ for solutions $u(x,t)$ of Problem $O$.

**Theorem 2.1.** *Under Condition $C'$, any solution $u(x,t)$ of Problem O for $(1.7)$ satisfies the estimate*

$$\hat{C}^{1,0}[u,\bar{Q}]=||u||_{\hat{C}^{1,0}(\overline{Q})}=|||u|^{\sigma+1}||_{C^{0,0}(\bar{Q})}+\sum_{i=1}^{N}||u_{x_i}||_{C^{0,0}(\overline{Q})}\leq M_1, \tag{2.1}$$

*where $p_0\,(2<p_0\leq p)$ and $M_1=M_1(q,p_0,\alpha,\ k,Q)$ are non-negative constants with $q=(q_0,q_1),\ k=(k_0,k_1,k_2)$.*

*Proof.* Suppose that (2.1) is not true. Then there exist sequences of functions $\{a_{ij}^m\}$, $\{b_i^m\}$, $\{c^m\}$, $\{f^m\}$ and $\{g^m(x)\}$, $\{d^m(x,t)\}$, $\{\sigma^m(x,t)\}$, $\{\tau^m(t,x)\}$, which satisfy Condition $C$ and the conditions in (1.10), such that $\{a_{ij}^m\}$, $\{b_i^m\}$, $\{c^m\}$, $\{f^m\}$ weakly converge to $a_{ij}^0$, $b_i^0$, $c^0$, $f^0$, and $\{g^m\}$, $\{d^m\}$, $\{\sigma^m\}$, $\{\tau^m\}$ uniformly converge to $g^0$, $d^0$, $\sigma^0$, $\tau^0$ in $\Omega$ or $S_2$ respectively, and the initial-boundary value problem

$$\sum_{i,j=1}^{n}a_{ij}^m u_{x_ix_j}+\sum_{i=1}^{n}b_i^m u_{x_i}+\tilde{c}^m u-Hu_t=f^m \ \text{ in } Q, \tag{2.2}$$

$$u^m(x,0)=g^m(x) \ \text{ on } \Omega, \tag{2.3}$$

$$lu^m(x,t)=\tau^m(x,t), \ \text{ i.e. } d^m\frac{\partial u^m}{\partial\nu}+\sigma^m u^m=\tau^m(x,t) \ \text{ on } S_2 \tag{2.4}$$

has a solution $u^m(x,t)$, such that $||u^m||_{\hat{C}^{1,0}(\overline{Q})}=H_m(m=1,2,...)$ is unbounded. There is no harm in assuming that $H_m\geq 1$, and $\lim_{m\to\infty}H_m=+\infty$. It is easy to see that $U^m=u^m/H_m$ is a solution of the initial-boundary value problem

$$\sum_{i,j=1}^{N}a_{ij}^m U_{x_ix_j}^m-HU_t^m=G^m, \ \ G^m=-\sum_{i=1}^{N}b_i^m U_{x_i}^m-\tilde{c}^m U^m+\frac{f^m}{H_m}, \tag{2.5}$$

$$U^m(x,0)=\frac{g^m(x)}{H_m}, \ \ x\in\Omega, \tag{2.6}$$

$$lU^m(x,t)=\frac{\tau^m}{H_m}, \ \text{ i.e. } d^m\frac{\partial U^m}{\partial\nu}+\sigma^m U^m=\frac{\tau^m}{H_m}, \ \ (x,t)\in S_2. \tag{2.7}$$

Noting that $L_p[\sum_{i=1}^N b_i^m U_{x_i}^m+c^m U^m]$ is bounded and using the result in Theorem 2.2 below, we can obtain the estimate

$$\begin{aligned}\hat{C}^{1,0}_{\beta,\beta/2}[U^m,\bar{Q}]&=||U^m||_{\hat{C}^{1,0}_{\beta,\beta/2}(\bar{Q})}\\ &=|||U^m|^{\sigma+1}||_{C^{0,0}_{\beta,\beta/2}(\bar{Q})}+\sum_{i=1}^{N}||U_{x_i}^m||_{C^{0,0}_{\beta,\beta/2}(\bar{Q})}\leq M_2,\end{aligned} \tag{2.8}$$

$$||U^m||_{\tilde{W}^{2,1}_{p_0}(Q)}\leq M_2=M_2(q,p_0,\alpha,k,Q), \ \ m=1,2,..., \tag{2.9}$$

where $\beta\,(0<\beta\leq\alpha)$, $p_0\,(2\leq p_0<p)$, $M_2=M_2(q,p_0,\alpha,k,Q)$ are non-negative constants. Hence from $\{U^m\},\{U_{x_i}^m\}$, we can choose a subsequence $\{U^{m_k}\}$ such that $\{U^{m_k}\}$, $\{U_{x_i}^{m_k}\}$ uniformly converge to $U^0,U_{x_i}^0$ in $\bar{Q}$ and $\{U_{x_ix_j}^{m_k}\},\{U_t^{m_k}\}$ weakly converge to $U_{x_ix_j}^0,U_t^0$ in $Q$ respectively, where $U^0$ is a solution of the initial-boundary value problem

$$\sum_{i,j=1}^{N}a_{ij}^0\tilde{U}_{x_ix_j}^0+\sum_{i=1}^{N}b_i^0\tilde{U}_{x_i}^0+\tilde{c}^0U^0-HU_t^0=0, \tag{2.10}$$

$$U^0(x,0)=0,\ \ x\in\Omega, \tag{2.11}$$

$$lU^0(x,t)=0,\ \text{ i.e. } d\frac{\partial U^0}{\partial\nu}+\sigma U^0=0,\ \ (x,t)\in S_2. \tag{2.12}$$

According to Theorem 1.1, we know $U^0(x,t)=0,(x,t)\in\overline{Q}$. However, from $||U^m||_{\hat{C}^{1,0}(\overline{Q})}=1$, there exists a point $(x^*,t^*)\in\overline{Q}$, such that $|U^0(x^*,t^*)|+\sum_{i=1}^N|U^0_{x_i}(x^*,t^*)|>0$. This contradiction proves that (2.1) is true. □

**Theorem 2.2.** *Under the same condition in Theorem* $2.1$*, any solution* $u(x,t)$ *of Problem* $O$ *satisfies the estimates*

$$||u||_{\hat{C}^{1,0}_{\beta,\beta/2}(\bar{Q})}\le M_3=M_3(q,p_0,\alpha,k,Q), \tag{2.13}$$

$$||u||_{\tilde{W}^{2,1}_{p_0}(Q)}\le M_4=M_4(q,p_0,\alpha,k,Q), \tag{2.14}$$

*where* $\beta\,(0<\beta\le\alpha),p_0\,(2\le p_0\le p),M_j\,(j=3,4)$ *are non-negative constants.*

*Proof.* First of all, we find a solution $\hat{u}(x,t)$ of the equation

$$\Delta\hat{u}-H\hat{u}_t=0 \tag{2.15}$$

with the initial-boundary conditions (1.8) and (1.9), which satisfies the estimate

$$||\hat{u}||_{C^{2,1}(\bar{Q})}\le M_5=M_5(q,p_0,\alpha,k,Q) \tag{2.16}$$

(see Chapter III of [4] and [2]). Thus the function

$$\tilde{u}(x,t)=u(x,t)-\hat{u}(x,t) \tag{2.17}$$

is a solution of the equation

$$\mathcal{L}\tilde{u}=\sum_{i,j=1}^N a_{ij}\tilde{u}_{x_ix_j}+\sum_{i=1}^N b_i\tilde{u}_{x_i}+\tilde{c}\tilde{u}-H\tilde{u}_t=\tilde{f}, \tag{2.18}$$

$$\tilde{u}(x,0)=0,\ \ x\in\Omega, \tag{2.19}$$

$$l\tilde{u}(x,t)=0,\ \ (x,t)\in S_2, \tag{2.20}$$

where $\tilde{f}=f-\mathcal{L}\hat{u}$. Introduce a local coordinate system $x=x(\xi)$ on the neighborhood $G$ of a surface $S_0\in\partial\Omega$ as follows:

$$x_i=h_i(\xi_1,...,\xi_{N-1})\xi_N+g_i(\xi_1,...,\xi_{N-1}),\ \ i=1,...,N, \tag{2.21}$$

where $\xi_N=0$ is just the surface $S_0:\ x_i=g_i(\xi_1,...,\xi_{N-1})\,(i=1,...,N)$, and

$$h_i(\xi)=\left.\frac{d_i(x)}{d(x)}\right|_{x_i=g_i(\xi)},\ \ i=1,...,N,\ \ d^2(x)=\sum_{i=1}^N d_i^2(x).$$

Then the boundary condition (2.20) can be reduced to the form

$$\frac{\partial\tilde{u}}{\partial\xi_N}+\tilde{\sigma}=0\ \text{ on }\ \xi_N=0, \tag{2.22}$$

where $\tilde{u} = \tilde{u}[x(\xi), t]$, $\tilde{\sigma} = \sigma[x(\xi), t]$. Secondly, we find a solution $v(x,t)$ of Problem $N$ for equation (2.15) with the boundary condition

$$\frac{\partial v}{\partial \xi_N} = \tilde{\sigma} \text{ on } \xi_N = 0, \tag{2.23}$$

which satisfies the estimate

$$||v||_{C^{2,1}(\bar{Q})} \le M_6 = M_6(q, p_0, \alpha, k, Q) < \infty. \tag{2.24}$$

It is seen that the function

$$V(x,t) = \tilde{u}\mathrm{e}^{v(x,t)} \tag{2.25}$$

is a solution of the initial-boundary value problem

$$\sum_{i,j=1}^{N} \tilde{a}_{ij} V_{\xi_i \xi_j} + \sum_{i=1}^{N} \tilde{b}_i V_{x_i} + \tilde{c}V - HV_t = \tilde{f}, \tag{2.26}$$

$$\frac{\partial V}{\partial \xi_N} = 0, \quad \xi_N = 0. \tag{2.27}$$

On the basis of Theorem 3.3, Chapter III, [4], we can derive the following estimates of $V(\xi, t)$:

$$||V||_{\hat{C}^{1,0}_{\beta,\beta/2}(\bar{Q})} \le M_7 = M_7(q, p_0, \alpha, k, Q), \tag{2.28}$$

$$||V||_{\tilde{W}^{2,1}_{p_0}(Q)} \le M_8 = M_8(q, p_0, \alpha, k, Q), \tag{2.29}$$

where $\beta\,(0<\beta\le\alpha), p_0\,(2\le p_0<p), M_j\;(j=7,8)$ are non-negative constants. Combining $(2.16)$, $(2.24)$, $(2.28)$ and (2.29), the estimates (2.13) and (2.14) are obtained. □

By using the similar method as in the proof of Theorem 2.1, we can prove the following theorem.

**Theorem 2.3.** *Suppose that equation* $(1.7)$ *satisfies Condition* $C'$. *Then any solution* $u(x,t)$ *of Problem* $O$ *satisfies the estimates*

$$\hat{C}^{1,0}_{\beta,\beta/2}[u, \bar{Q}] = ||u||_{\hat{C}^{1,0}_{\beta,\beta/2}(\bar{Q})} \le M_9(k_1 + k_2), \tag{2.30}$$

$$||u||_{\tilde{W}^{2,1}_{p_0}(Q)} \le M_{10}(k_1 + k_2), \tag{2.31}$$

*where* $\beta(0 < \beta \le \alpha)$, $p_0\,(2 \le p_0 < p), M_j = M_j(q, p_0, \alpha, k_0, Q)\,(j = 9, 10)$ *are non-negative constants.*

## 3 Solvability of Initial-Oblique Derivative Problems for Parabolic Equations

We first consider a special equation of (1.7):

$$\begin{gathered}\Delta u - Hu_t = g_m(x, t, u, D_x u, D_x^2 u),\\ g_m = \Delta u - \sum_{i,j=1}^{N} a_{ijm} u_{x_i x_j} - \sum_{i=1}^{N} b_{im} u - \tilde{c}_m u + f_m \text{ in } Q,\end{gathered} \tag{3.1}$$

with $\Delta u=\sum_{i=1}^N \partial^2 u/\partial x_i^2$, $\Lambda=2\inf_Q\sum_{i=1}^N a_{ii}/(2N-1)$, and the coefficients

$$\begin{aligned}
&a_{ijm}=\begin{cases} a_{ij}/\Lambda, \\ \delta_{ij}/\Lambda, \end{cases} \quad b_{im}=\begin{cases} b_i/\Lambda, \\ 0, \end{cases} \quad i,j=1,...,N, \\
&\tilde{c}_m=\begin{cases} \tilde{c}/\Lambda, \\ 0, \end{cases} \quad H=\begin{cases} 1/\Lambda, \\ 1/\Lambda, \end{cases} \quad f_m=\begin{cases} f/\Lambda & \text{in } Q_m, \\ 0 & \text{in } \{\mathbf{R}^N\times I\}\backslash Q_m, \end{cases}
\end{aligned} \tag{3.2}$$

where $Q_m=\{(x,t)\in Q\,|\,\mathrm{dist}((x,t),\partial Q)\geq 1/m\}$ for a positive integer $m$, $\delta_{ii}=1$, $\delta_{ij}=0\,(i\neq j,\, i,j=1,...,N)$. In particular, the linear case of equation (3.1) can be written as

$$\begin{aligned}
\Delta u-Hu_{\Lambda t}=g_m(x,t,u,D_xu,D_x^2u),\quad g_m=\sum_{i,j=1}^N[\delta_{ij}-a_{ijm}(x,t)]u_{x_ix_j} \\
-\sum_{i=1}^N b_{im}(x,t)u_{x_i}-c_m(x,t)u+f_m(x,t)\ \text{ in } Q.
\end{aligned} \tag{3.3}$$

In the following, we will give the representation of solutions of Problem $O$ for equation (3.1).

**Theorem 3.1.** *Under Condition $C'$, if $u(x,t)$ is a solution of Problem $O$ for equation $(3.1)$, then $u(x,t)$ can be expressed in the form*

$$\begin{aligned}
&u(x,t)=U(x,t)+\hat{V}(x,t)=U(x,t)+v_0(x,t)+v(x,t), \\
&v(x,t)=\tilde{H}\rho=\int_{Q_0}G(x,t,\zeta,\tau)\rho(\zeta,\tau)d\sigma_\zeta d\tau, \\
&G=\begin{cases} [\Lambda(t-\tau)]^{-N/2}\exp[|x-\zeta|^2/4\Lambda(t-\tau)], & t>\tau, \\ 0,\ \ t\leq\tau,\ \ \text{except } t-\tau=|x-\zeta|=0. \end{cases}
\end{aligned} \tag{3.4}$$

*In (3.4), $\rho(x,t)=\Delta u-Hu_t=\Delta u-u_{\Lambda t}=g_m$. $\hat{V}(x,t)$ is a solution of Problem $D$ for $(3.1)$ in $Q_0=\Omega_0\times I$ with the initial-boundary condition $\hat{V}(x,t)=0$ on $\partial Q_0$, where $\Omega_0=\{|x|<R\}$ for a large number $R$ such that $\Omega_0\supset\overline{\Omega}$. $U(x,t)$ is a solution of Problem $\tilde{P}$ for $LU=\Delta U-U_{\Lambda t}=0$ in $Q$ with the initial-boundary condition $(3.12)-(3.13)$ below. $\hat{V}(x,t)$ and $U(x,t)$ satisfy the estimates*

$$\begin{aligned}
C^{1,0}_{\beta,\beta/2}[U,\overline{Q}]+||U||_{\tilde{W}_2^{2,1}(Q)}\leq M_{11}, \\
C^{1,0}_{\beta,\beta/2}[\hat{V},\overline{Q_0}]+||\hat{V}||_{\tilde{W}_2^{2,1}(Q_0)}\leq M_{12},
\end{aligned} \tag{3.5}$$

*where $\beta(0<\beta\leq\alpha)$, $M_j=M_j(q,p_0,\alpha,k,Q_m)\,(j=11,12)$ are non-negative constants, $q=(q_0,q_1)$, $k=(k_0,k_1,k_2)$.*

*Proof.* It is easy to see that the solution $u(x,t)$ of Problem $O$ for equation (3.1) can be expressed by the form (3.4). Since $a_{ijm}=0\,(i\neq j)$, $b_{im}=0$, $c_m=0$, $f_m(x,t)=0$ in $\{\mathbf{R}^N\times I\}\backslash Q_m$ and $\hat{V}(x,t)$ is a solution of Problem $D$ for (3.1) in $Q_0$, we can see that $\hat{V}(x,t)$ in $\hat{Q}_{2m}=\overline{Q}\backslash Q_{2m}$ satisfies the estimate

$$C^{2,1}[\hat{V}(x,t),\hat{Q}_{2m}]\leq M_{13}=M_{13}(q,p_0,\alpha,k,Q_m).$$

On the basis of Theorem 2.3, we can see that $U(x,t)$ satisfies the first estimate in (3.5), and then $\hat{V}(x,t)$ satisfies the second estimate in (3.5). □

**Theorem 3.2.** *If equation* $(1.7)$ *satisfies Condition* $C'$*, then Problem O for* $(3.3)$ *has a solution* $u(x,t)$*.*

*Proof.* In order to prove the existence of solutions of Problem $O$ for the nonlinear equation (3.1) by using the Larey-Schauder theorem, we introduce the equation with the parameter $h \in [0,1]$

$$\Delta u - u_{\Lambda t} = hg_m(x,t,u,D_x u, D_x^2 u) \ \text{ in } \ Q. \tag{3.6}$$

Denote by $B_M$ a bounded open set in the Banach space $B = \hat{W}_2^{2,1}(Q) = \hat{C}^{1,0}_{\beta,\beta/2}(\overline{Q}) \cap \tilde{W}_2^{2,1}(Q)$ for $0 < \beta \le \alpha$, the elements of which are real functions $V(x,t)$ satisfying the inequalities

$$||V||_{\hat{W}_2^{2,1}(Q)} = \hat{C}^{1,0}_{\beta,\beta/2}[V,\bar{Q}] + ||V||_{\tilde{W}_2^{2,1}(Q)} < M_{14} = M_3 + M_4 + 1, \tag{3.7}$$

in which $\tilde{W}_2^{2,1}(Q) = W_2^{2,0}(Q) \cap W_2^{0,1}(Q)$, $M_3, M_4$ are the non-negative constants as stated in (2.13) and (2.14). We choose any function $\tilde{V}(x,t) \in \overline{B_M}$ and substitute it into the appropriate positions on the right hand side of (3.6), and then we make an integral $\tilde{v}(x,t) = \tilde{H}\rho$ as follows:

$$\tilde{v}(x,t) = \tilde{H}\rho, \ \ \rho(x,t) = \Delta\tilde{V} - \tilde{V}_{\Lambda t}. \tag{3.8}$$

Next we find a solution $\hat{v}(x,t)$ of the initial-boundary value problem in $Q_0$:

$$\Delta\hat{v}_0 - \hat{v}_{0\Lambda t} = 0 \ \text{ on } \ Q_0, \tag{3.9}$$

$$\hat{v}(x,t) = -\tilde{v}(x,t) \ \text{ on } \ \partial Q_0, \tag{3.10}$$

and denote by $\hat{V}(x,t) = \hat{v}(x,t) + \tilde{v}(x,t)$ the solution of the corresponding Problem $D$ in $Q_0$. Moreover, on the basis of the result in Chapter III of [4] and [2], we can find a solution $U(x,t)$ of the corresponding Problem $\tilde{O}$ in $Q$:

$$\Delta U - U_{\Lambda t} = 0 \ \text{ on } \ Q, \tag{3.11}$$

$$U(x,0) = g(x) - \hat{V}(x,0) \ \text{ on } \ \Omega, \tag{3.12}$$

$$\frac{\partial U}{\partial \nu} + \sigma(x,t)U = \tau(x,t) - \frac{\partial \hat{V}}{\partial \nu} + \sigma(x,t)\hat{V} \ \text{ on } \ S_2. \tag{3.13}$$

Now we discuss the equation

$$\Delta V - V_{\Lambda t} = hg_m(x,t,u,D_x u, D_x^2 U + D_x^2 \hat{V}), \ 0 \le h \le 1, \tag{3.14}$$

where $u = U + \hat{V}$. By Condition $C$, applying the principle of contracting mapping, we can find a unique solution $V(x,t)$ of Problem $D$ for equation (3.14) in $Q_0$ satisfying the initial-boundary condition

$$V(x,t) = 0 \ \text{ on } \ \partial Q_0. \tag{3.15}$$

Here we mention that due to Section 2, Chapter I, [4] and the result in [3], we can use the principle of contracting mapping. If we do not have the conditions and results, it is impossible to use the principle. Set $u(x,t) = U(x,t) + \hat{V}(x,t)$, where the relation between $U$ and $\tilde{V}$ is the same as that between $u$ and $\tilde{V}$, and denote by $V = S(\tilde{V},h)$ and $u = S_1(\tilde{V},h)\,(0 \le h \le 1)$ the mappings from $\tilde{V}$ onto $V$ and $u$ respectively. Furthermore, if $V(x,t)$ is a solution of Problem $D$ in $Q_0$ for the equation

$$\Delta V - V_{\Lambda t} = hg_m(x,t,u,D_x u, D_x^2(U+V)), \ \ 0 \le h \le 1, \tag{3.16}$$

where $u = S_1(V,h)$, then from Theorem 3.1, the solution $V(x,t)$ of Problem $D$ for (3.16) satisfies the estimate (3.7), and consequently $V(x,t) \in B_M$. Set $B_0 = B_M \times [0,1]$. In the following, we shall verify that the mapping $V = S(\tilde{V},h)$ satisfies the three conditions of Leray-Schauder theorem.

1) For every $h \in [0,1],\ V = S(\tilde{V},h)$ continuously maps the Banach space $B$ into itself, and is completely continuous on $B_M$. Besides, for every function $\tilde{V}(x,t) \in \overline{B_M},\ S(\tilde{V},h)$ is uniformly continuous with respect to $h \in [0,1]$.

In fact, we arbitrarily choose $\tilde{V}_l(x,t) \in \overline{B_M}\,(l = 1,2,...)$. It is clear that from $\{\tilde{V}_l(x,t)\}$ there exists a subsequence $\{\tilde{V}_{l_k}(x,t)\}$ such that $\{\tilde{V}_{l_k}(x,t)\}$, $\{\tilde{V}_{l_k x_i}(x,t)\}$ $(i = 1,...,N)$ and corresponding functions $\{U_{l_k}(x,t)\}$, $\{U_{l_k x_i}(x,t)\}$, $\{u_{l_k}(x,t)\}$, $\{u_{l_k x_i}(x,t)\}$ $(i = 1,...,N)$ uniformly converge to $\tilde{V}_0(x,t)$, $\tilde{V}_{0x_i}(x,t)$, $U_0(x,t)$, $U_{0x_i}(x,t)$, $u_0(x,t)$, $u_{0x_i}(x,t)$ $(i = 1,...,N)$ in $\overline{Q_0}$, $\overline{Q}$ respectively, in which $u_{l_k} = S_1(\tilde{V}_{l_k},h)$, $u_0 = S_1(\tilde{V}_0,h)$. We can find a solution $V_0(x,t)$ of Problem $D$ for the equation

$$\Delta V_0 - V_{0\Lambda t} = hg_m(x,t,u_0,D_x u_0, D_x^2 U_0 + D_x^2 \hat{V}_0), \ \ 0 \le h \le 1 \ \text{ in } Q_0. \tag{3.17}$$

From $V_{l_k} = S(\tilde{V}_{l_k},h)$ and $V_0 = S(\tilde{V}_0,h)$, we have

$$\Delta(V_{l_k} - V_0) - (V_{l_k} - V_0)_{\Lambda t} = h[g_m(x,t,u_{l_k},D_x u_{l_k}, D_x^2 U_{l_k} + D_x^2 \hat{V}_{l_k})$$

$$-g_m(x,t,u_{l_k},D_x u_{l_k}, D_x^2 U_{l_k} + D_x^2 \hat{V}_0) + C_{l_k}(x,t)], \ \ 0 \le h \le 1,$$

where

$$C_{l_k}(z,t) = g_m(x,t,u_{l_k},D_x u_{l_k}, D_x^2 U_{l_k} + D_x^2 \hat{V}_0)$$

$$-g_m(x,t,u_0,D_x u^0, D_x^2 U_0 + D_x^2 \hat{V}_0), \ \ (x,t) \in Q_0.$$

Later we shall prove that

$$L_2[C_{l_k}(x,t),\overline{Q_0}] \to 0 \ \text{ as } \ k \to \infty. \tag{3.18}$$

Moreover, according to Theorem 2.3, we can derive that

$$||V_{l_k} - V_0||_{\hat{W}_2^{2,1}(Q_0)} \le M_{15} L_2[C_{l_k},\overline{Q_0}],$$

where $M_{15} = M_{15}(q,p_0,\alpha,k_0,Q_m)$ is a non-negative constant, and hence $||V_{l_k} - V_0||_{\hat{W}_2^{2,1}(Q_0)} \to 0$ as $k \to \infty$. Thus from $\{\tilde{V}_{l_k}(x,t) - \tilde{V}_0(x,t)\}$, there exists a subsequence, denoted, for convenience, by $\{\tilde{V}_{l_k}(x,t) - \tilde{V}_0(x,t)\}$, such that $||V_{l_k}(x,t) - V_0(x,t)||_{\hat{W}_2^{2,1}(Q_0)} = \hat{C}^{1,0}_{\beta,\beta/2}[V_{l_k}(x,t) - V_0(x,t),\overline{Q_0}] + ||V_{l_k}(x,t) - V_0(x,t)||_{\tilde{W}_2^{2,1}(Q_0)} \to 0$ as $k \to \infty$. From this we can obtain that the corresponding subsequence $\{u_{l_k}(x,t) - u_0(x,t)\} = \{S_1(\tilde{V}_{l_k},h) - S_1(\tilde{V}_0,h)\}$ possesses the property: $||u_{l_k}(x,t) - u_0(x,t)||_{\hat{W}_2^{2,1}(Q)} \to 0$ as $k \to \infty$. This shows the complete continuity of $V = S(\tilde{V},h)\,(0 \le h \le 1)$ in $\overline{B_M}$. By using the similar method, we can prove that

$V = S(\tilde{V}, h)\,(0 \le h \le 1)$ continuously maps $\overline{B_M}$ into $B$, and $V = S(\tilde{V}, h)$ is uniformly continuous with respect to $h \in [0,1]$ for $\tilde{V} \in \overline{B_M}$.

2) For $h = 0$, from Theorem 2.2 and (3.7), it is clear that $V = S(\tilde{V}, 0) \in B_M$.

3) From Theorem 2.2 and (3.7), we see that $V = S(\tilde{V}, h)(0 \le h \le 1)$ does not have a solution $u(x,t)$ on the boundary $\partial B_M = \overline{B_M} \backslash B_M$.

Hence by the Leray-Schauder theorem, we know that Problem $D_0$ for equation (3.6) with $h=1$ has a solution $V(z,t) \in B_M$, and then Problem $O$ of equation (3.6) with $h = 1$, i.e. (3.1) has a solution $u(x,t)=S_1(\tilde{V},h)=U(x,t)+V(x,t)=U(x,t)+\hat{v}(x,t)+\tilde{v}(x,t) \in B$.

Finally, we verify (3.18). In fact, by Condition $C'$ and the above discussion, we can choose, from $\{C_{l_k}(x,t)\}$, a subsequence denoted by $\{C_{l_k}(x,t)\}$ again, such that $\{C_{l_k}(x,t)\}$ converges 0 for almost every point in $Q_0$. Hence for two sufficiently small positive numbers $\varepsilon_1, \varepsilon_2$, there exist a subset $Q_*$ in $Q_0$ and a positive number $K_0$, such that $\operatorname{mes} Q_* < \varepsilon_1$ and $|C_{l_k}| < \varepsilon_2,\ (x,t) \in \overline{Q}_0 \backslash Q_*$ as $k > K_0$. According to the Hölder inequality and Minkowski inequality, we have

$$
\begin{aligned}
& L_2[C_{l_k}, Q_*] + L_2[C_{l_k}, \overline{Q_0} \backslash Q_*] \\
& \le L_{p_1}[C_{l_k}, D_*] L_{p_2}[1, D_*] + \varepsilon_2\,(\mathrm{mes}\,\mathrm{Q}_0)^{1/2} \\
& \le \varepsilon_1^{1/p_2} M_{16} + \varepsilon_2 (\mathrm{mes}\,\mathrm{Q}_0)^{1/2} = \varepsilon,
\end{aligned}
$$

where $p_2$ is a sufficiently large positive constant, $p_1 = 2p_2/(p_2 - 2)$ is a positive constant near 2, and $M_{16} = \sup_{1 \le k < \infty} L_{p_1}[C_{l_k}, Q_*]$ is a constant. Provided that $\varepsilon_1, \varepsilon_2$ are small enough, $\varepsilon$ can be sufficiently small. This shows that (3.18) is true. □

**Theorem 3.3.** *Under the same condition in Theorem* $3.2$*, Problem* $O$ *for equation* $(1.7)$ *has a solution.*

*Proof.* By Theorems 2.2 and 3.2, Problem $O$ for equation (3.1) possesses a solution $u_m(x,t)$ that satisfies the estimates (2.30) and (2.31)$(m = 1, 2, \ldots)$. Thus, we can choose a subsequence $\{u_{m_k}(x,t)\}$, such that $\{u_{m_k}(x,t)\}$, $\{u_{m_k x_i}(x,t)\}\,(i = 1, \ldots, N)$ in $\overline{Q}$ uniformly converge to $u_0(x,t)$, $u_{0x_i}(x,t)\,(i = 1, \ldots, N)$, respectively. Obviously, $u_0(x,t)$ satisfies the initial-boundary conditions of Problem $O$. On the basis of principle of compactness of solutions for equation (3.1) (see Theorem 4.6, Chapter I, [4]), we can see that $u_0(x,t)$ is a solution of Problem $O$ for (1.7). □

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# Some solutions of generalized Rassias's equation

**Anvar Hasanov**

Institute of Mathematics, Uzbek Academy of Sciences,
29, F. Hodjaev street, Tashkent 700125, Uzbekistan

E-mail: mathinst@uzsci.net, anvarhasanov@yahoo.com

**Abstract**

*In the field of* $\mathbb{R}_+^3 = \{(x,y,z) : x > 0, y > 0, z > 0\}$ *the generalized Rassias's equation* $R(u) \equiv y^m z^k u_{xx} - x^n z^k u_{yy} + x^n y^m u_{zz} = 0,$ $m,n,k = const > 0,$ *is considered. By means of a change of variables the generalized Rassias's equation to be reduced to a system of hypergeometric equations for the function of Lauricella of three variables. Eight linearly independent particular solutions of the system of hypergeometric equations are found. Properties of found particular solutions are studied by virtue of decomposition of the hypergeometric function of Lauricella.*

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**Key Words and Phrases**. Degenerating hyperbolic equation, particular solution, singular partial differential equation.

## 1 INTRODUCTION

In paper [1] studies the equation

$$R(u) \equiv k(z)\left(u_{xx} - u_{yy}\right) + u_{zz} + \lambda(x,y,z)u = f(x,y,z), \tag{1.1}$$

$$k(z) > 0, for\, z > 0,$$
$$k(z) < 0, for\, z < 0,$$

in a simply connected bounder domain $D \subset \mathbb{R}^3$. There are some works, for example: [2-4] in which the some modifications of equation (1.1) are considered. We shall consider the equation (1.1) in the case of when $k(z) = 1, \lambda(x,y,z) = f(x,y,z) = 0$ and we shall construct particular solutions of the equation (1.1). Also, we study the generalized Rassias's equation which degenerates in each hyper plane of the space $\mathbb{R}_+^3 = \{(x,y,z) : x > 0, y > 0, z > 0\}$. For the generalized Rassias's equation, we find eight linearly independent particular solutions. Using decompositions of hypergeometric function of Lauricella in a series on products of hypergeometric function of Gauss, we investigate behavior of the found particular solutions at $\sigma \to 0$.

## 2. PARTICULAR SOLUTIONS Of RASSIAS'S EQUATION

In the domain, $\mathbb{R}^3 = \{(x, y, z) : -\infty < x < +\infty,\ -\infty < y < +\infty,\ -\infty < z < +\infty,\ \}$ we shall consider the equation

$$L(u) \equiv u_{xx} - u_{yy} + u_{zz} = 0. \tag{2.1}$$

The solution of the equation (2.1) we shall search in the form of

$$u = \omega(\sigma), \tag{2.2}$$

Where $\omega(\sigma)$ is unknown function and

$$\sigma = (x - x_0)^2 - (y - y_0)^2 + (z - z_0)^2. \tag{2.3}$$

Let's calculate derivatives

$$\begin{gathered} u_x = \omega_\sigma \sigma_x,\ u_y = \omega_\sigma \sigma_y,\ u_z = \omega_\sigma \sigma_z, \\ u_{xx} = \omega_{\sigma\sigma}\sigma_x^2 + \omega_\sigma \sigma_{xx},\ u_{yy} = \omega_{\sigma\sigma}\sigma_y^2 + \omega_\sigma \sigma_{yy},\ u_{zz} = \omega_{\sigma\sigma}\sigma_z^2 + \omega_\sigma \sigma_{zz}. \end{gathered} \tag{2.4}$$

Substituting (2.4) in to the (2.1) we get

$$\left[\sigma_x^2 - \sigma_y^2 + \sigma_z^2\right]\omega_{\sigma\sigma} + \left[\sigma_{xx} - \sigma_{yy} + \sigma_{zz}\right]\omega_\sigma = 0. \tag{2.5}$$

We calculate derivatives

$$\sigma_x = 2(x - x_0),\ \sigma_y = -2(y - y_0), \sigma_z = 2(z - z_0), \sigma_{xx} = 2,\ \sigma_{yy} = -2, \sigma_{zz} = 2. \tag{2.6}$$

Substituting (2.6) in to the equation (2.5), we have

$$2\sigma\omega_{\sigma\sigma} + 3\omega_\sigma = 0. \tag{2.7}$$

The equations (2.7) has the following solution

$$\omega = c_1 \sigma^{-\frac{1}{2}} + c_2,\ c_1, c_2 - const. \tag{2.8}$$

## 3. SOME PARTICULAR SOLUTIONS OF GENERALIZED RASSIAS'S EQUATION

We consider in the domain $\mathbb{R}_+^3 = \{(x, y, z) : x > 0, y > 0, z > 0\}$ a generalized Rassias's equation

$$R(u) \equiv y^m z^k u_{xx} - x^n z^k u_{yy} + x^n y^m u_{zz} = 0,\ m, n, k = const > 0. \tag{3.1}$$

The solution of the equation (3.1) we search in the form of

$$u = P\omega(\xi, \eta, \zeta), \tag{3.2}$$

where

$$P = \sigma^{-\alpha-\beta-\gamma-\frac{1}{2}}, \tag{3.3}$$

$$\xi = \frac{\sigma-\sigma_1}{\sigma},\ \eta = \frac{\sigma-\sigma_2}{\sigma},\ \zeta = \frac{\sigma-\sigma_3}{\sigma},\ \alpha = \frac{n}{2(n+2)},\ \beta = \frac{m}{2(m+2)},\ \gamma = \frac{k}{2(k+2)}, \tag{3.4}$$

$$\left.\begin{matrix}\sigma\\ \sigma_1\\ \sigma_2\\ \sigma_3\end{matrix}\right\} = \left(\frac{2}{n+2}x^{\frac{n+2}{2}} \begin{smallmatrix}-\\+\\-\\-\end{smallmatrix} \frac{2}{n+2}x_0^{\frac{n+2}{2}}\right)^2 - \left(\frac{2}{m+2}y^{\frac{m+2}{2}} \begin{smallmatrix}-\\-\\+\\-\end{smallmatrix} \frac{2}{m+2}y_0^{\frac{m+2}{2}}\right)^2 + \left(\frac{2}{k+2}z^{\frac{k+2}{2}} \begin{smallmatrix}-\\-\\-\\+\end{smallmatrix} \frac{2}{k+2}z_0^{\frac{k+2}{2}}\right)^2. \tag{3.5}$$

Substituting (3.2) in to the (3.1), we have

$$A_1\omega_{\xi\xi} + A_2\omega_{\eta\eta} + A_3\omega_{\zeta\zeta} + B_1\omega_{\xi\eta} + B_2\omega_{\xi\zeta} + B_3\omega_{\eta\zeta} + C_1\omega_{\xi} + C_2\omega_{\eta} + C_3\omega_{\zeta} + D\omega = 0, \tag{3.6}$$

where

$$A_1 = P\left[y^m z^k \xi_x^2 - x^n z^k \xi_y^2 + x^n y^m \xi_z^2\right],$$

$$A_2 = P\left[y^m z^k \eta_x^2 - x^n z^k \eta_y^2 + x^n y^m \eta_z^2\right],$$

$$A_3 = P\left[y^m z^k \zeta_x^2 - x^n z^k \zeta_y^2 + x^n y^m \zeta_z^2\right],$$

$$B_1 = 2P\left[y^m z^k \xi_x\eta_x - x^n z^k \xi_y\eta_y + x^n y^m \xi_z\eta_z\right],$$

$$B_2 = 2P\left[y^m z^k \xi_x\zeta_x - x^n z^k \xi_y\zeta_y + x^n y^m \xi_z\zeta_z\right],$$

$$B_3 = 2P\left[y^m z^k \eta_x\zeta_x - x^n z^k \eta_y\zeta_y + x^n y^m \eta_z\zeta_z\right],$$

$$C_1 = 2y^m z^k P_x\xi_x - 2x^n z^k P_y\xi_y + 2x^n y^m P_z\xi_z + y^m z^k P\xi_{xx} - x^n z^k P\xi_{yy} + x^n y^m P\xi_{zz},$$

$$C_2 = 2y^m z^k P_x\eta_x - x^n z^k 2P_y\eta_y + 2x^n y^m P_z\eta_z + y^m z^k P\eta_{xx} - x^n z^k P\eta_{yy} + x^n y^m P\eta_{zz},$$

$$C_3 = 2y^m z^k P_x\zeta_x - x^n z^k 2P_y\zeta_y + 2x^n y^m P_z\zeta_z + y^m z^k P\zeta_{xx} - x^n z^k P\zeta_{yy} + x^n y^m P\zeta_{zz},$$

$$D = y^m z^k P_{xx} - x^n z^k P_{yy} + x^n y^m P_{zz}.$$

After elementary evaluations, we find

$$A_1 = -\frac{4Px^n y^m z^k}{\sigma} x^{-\frac{n+2}{2}} x_0^{\frac{n+2}{2}} \xi(1-\xi), \tag{3.7}$$

$$A_2 = -\frac{4Px^n y^m z^k}{\sigma} y^{-\frac{m+2}{2}} y_0^{\frac{m+2}{2}} \eta(1-\eta), \tag{3.8}$$

$$A_3 = -\frac{4Px^n y^m z^k}{\sigma} z^{-\frac{k+2}{2}} z_0^{\frac{k+2}{2}} \zeta(1-\zeta), \tag{3.9}$$

$$B_1 = \frac{4Px^n y^m z^k}{\sigma} y^{-\frac{m+2}{2}} y_0^{\frac{m+2}{2}} \xi\eta + \frac{4Px^n y^m z^k}{\sigma} x^{-\frac{n+2}{2}} x_0^{\frac{n+2}{2}} \xi\eta, \tag{3.10}$$

$$B_2 = \frac{4Px^n y^m z^k}{\sigma} x^{-\frac{n+2}{2}} x_0^{\frac{n+2}{2}} \xi\zeta + \frac{4Px^n y^m z^k}{\sigma} z^{-\frac{k+2}{2}} z_0^{\frac{k+2}{2}} \xi\zeta, \tag{3.11}$$

$$B_3 = \frac{4Px^n y^m z^k}{\sigma} y^{-\frac{m+2}{2}} y_0^{\frac{m+2}{2}} \eta\zeta + \frac{4Px^n y^m z^k}{\sigma} z^{-\frac{k+2}{2}} z_0^{\frac{k+2}{2}} \eta\zeta, \tag{3.12}$$

$$\begin{aligned} C_1 = &-\frac{4Px^n y^m z^k}{\sigma} x^{-\frac{n+2}{2}} x_0^{\frac{n+2}{2}} \left[ 2\alpha - \left( \alpha + \beta + \gamma + \frac{1}{2} + \alpha + 1 \right) \xi \right] \\ &+ \frac{4Px^n y^m z^k}{\sigma} y^{-\frac{m+2}{2}} y_0^{\frac{m+2}{2}} \beta\xi + \frac{4Px^n y^m z^k}{\sigma} z^{-\frac{k+2}{2}} z_0^{\frac{k+2}{2}} \gamma\xi, \end{aligned} \tag{3.13}$$

$$\begin{aligned} C_2 = &\frac{4Px^n y^m z^k}{\sigma} x^{-\frac{n+2}{2}} x_0^{\frac{n+2}{2}} \alpha\eta - \frac{4Px^n y^m z^k}{\sigma} y^{-\frac{m+2}{2}} y_0^{\frac{m+2}{2}} \left[ 2\beta - \left( \alpha + \beta + \gamma + \frac{1}{2} + \beta + 1 \right) \eta \right] \\ &+ \frac{4Px^n y^m z^k}{\sigma} z^{-\frac{k+2}{2}} z_0^{\frac{k+2}{2}} \gamma\eta, \end{aligned} \tag{3.14}$$

$$\begin{aligned} C_3 = &\frac{4Px^n y^m z^k}{\sigma} x^{-\frac{n+2}{2}} x_0^{\frac{n+2}{2}} \alpha\zeta + \frac{4Px^n y^m z^k}{\sigma} y^{-\frac{k+2}{2}} y_0^{\frac{k+2}{2}} \beta\zeta \\ &- \frac{4Px^n y^m z^k}{\sigma} z^{-\frac{m+2}{2}} z_0^{\frac{m+2}{2}} \left[ 2\gamma - \left( \alpha + \beta + \gamma + \frac{1}{2} + \gamma + 1 \right) \zeta \right], \end{aligned} \tag{3.15}$$

$$\begin{aligned} D = &\frac{4Px^n y^m z^k}{\sigma} x^{-\frac{n+2}{2}} x_0^{\frac{n+2}{2}} \left( \alpha + \beta + \gamma + \frac{1}{2} \right) \alpha + \frac{4Px^n y^m z^k}{\sigma} y^{-\frac{m+2}{2}} y_0^{\frac{m+2}{2}} \left( \alpha + \beta + \gamma + \frac{1}{2} \right) \beta \\ &+ \frac{4Px^n y^m z^k}{\sigma} z^{-\frac{k+2}{2}} z_0^{\frac{k+2}{2}} \left( \alpha + \beta + \gamma + \frac{1}{2} \right) \gamma. \end{aligned} \tag{3.16}$$

Substituting (3.7) - (3.16) in to the (3.6) we get

$$\begin{aligned} &-\frac{4Px^n y^m z^k}{\sigma} x^{-\frac{n+2}{2}} x_0^{\frac{n+2}{2}} \\ &. \left\{ \begin{aligned} &\xi(1-\xi)\omega_{\xi\xi} - \xi\eta\omega_{\xi\eta} - \xi\zeta\omega_{\xi\zeta} + \left\{ 2\alpha - \left( \alpha + \beta + \gamma + \frac{1}{2} + \alpha + 1 \right) \xi \right\} \omega_\xi \\ &-\alpha\eta\omega_\eta - \alpha\zeta\omega_\zeta - \left( \alpha + \beta + \gamma + \frac{1}{2} \right) \alpha\omega \end{aligned} \right\} \\ &-\frac{4Px^n y^m z^k}{\sigma} y^{-\frac{m+2}{2}} y_0^{\frac{m+2}{2}} \\ &. \left\{ \begin{aligned} &\eta(1-\eta)\omega_{\eta\eta} - \xi\eta\omega_{\xi\eta} - \eta\zeta\omega_{\eta\zeta} + \left[ 2\beta - \left( \alpha + \beta + \gamma + \frac{1}{2} + \beta + 1 \right) \eta \right] \omega_\eta \\ &-\beta\xi\omega_\xi - \beta\zeta\omega_\zeta - \left( \alpha + \beta + \gamma + \frac{1}{2} \right) \beta\omega \end{aligned} \right\} \end{aligned} \tag{3.17}$$

$$-\frac{4Px^n y^m z^k}{\sigma} z^{-\frac{k+2}{2}} z_0^{\frac{k+2}{2}}$$

$$\cdot\left\{\begin{array}{l}\zeta(1-\zeta)\omega_{\zeta\zeta}-\xi\zeta\omega_{\xi\zeta}-\eta\zeta\omega_{\eta\zeta}+\left[2\gamma-\left(\alpha+\beta+\gamma+\frac{1}{2}+\gamma+1\right)\zeta\right]\omega_{\zeta}\\ -\gamma\xi\omega_{\xi}-\gamma\eta\omega_{\eta}-\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\gamma\omega\end{array}\right\}=0.$$

Solutions of the system of hypergeometric equations

$$\begin{cases}\xi(1-\xi)\omega_{\xi\xi}-\xi\eta\omega_{\xi\eta}-\xi\zeta\omega_{\xi\zeta}+\left\{2\alpha-\left(\alpha+\beta+\gamma+\frac{1}{2}+\alpha+1\right)\xi\right\}\omega_{\xi}-\alpha\eta\omega_{\eta}-\alpha\zeta\omega_{\zeta}\\ \qquad -\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\alpha\omega=0\\ \eta(1-\eta)\omega_{\eta\eta}-\xi\eta\omega_{\xi\eta}-\eta\zeta\omega_{\eta\zeta}+\left[2\beta-\left(\alpha+\beta+\gamma+\frac{1}{2}+\beta+1\right)\eta\right]\omega_{\eta}-\beta\xi\omega_{\xi}-\beta\zeta\omega_{\zeta}\\ \qquad -\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\beta\omega=0\\ \zeta(1-\zeta)\omega_{\zeta\zeta}-\xi\zeta\omega_{\xi\zeta}-\eta\zeta\omega_{\eta\zeta}+\left[2\gamma-\left(\alpha+\beta+\gamma+\frac{1}{2}+\gamma+1\right)\zeta\right]\omega_{\zeta}-\gamma\xi\omega_{\xi}-\gamma\eta\omega_{\eta}\\ \qquad -\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\gamma\omega=0,\end{cases} \tag{3.18}$$

also satisfies to the equation (3.17). The system of hypergeometric equations (3.18) has eight linearly independent solutions ([5], p. 117-118)

$$\omega_1(\xi,\eta,\zeta)=F_A^{(3)}\left(\alpha+\beta+\gamma+\frac{1}{2};\alpha,\beta,\gamma;2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\right), \tag{3.19}$$

$$\omega_2(\xi,\eta,\zeta)=\xi^{1-2\alpha}F_A^{(3)}\left(-\alpha+\beta+\gamma+\frac{3}{2};1-\alpha,\beta,\gamma;2-2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\right), \tag{3.20}$$

$$\omega_3(\xi,\eta,\zeta)=\eta^{1-2\beta}F_A^{(3)}\left(\alpha-\beta+\gamma+\frac{3}{2};\alpha,1-\beta,\gamma;2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\right), \tag{3.21}$$

$$\omega_4(\xi,\eta,\zeta)=\zeta^{1-2\gamma}F_A^{(3)}\left(\alpha+\beta-\gamma+\frac{3}{2};\alpha,\beta,1-\gamma;2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\right), \tag{3.22}$$

$$\omega_5(\xi,\eta,\zeta)=\xi^{1-2\alpha}\eta^{1-2\beta}F_A^{(3)}\left(-\alpha-\beta+\gamma+\frac{5}{2};1-\alpha,1-\beta,\gamma;2-2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\right), \tag{3.23}$$

$$\omega_6(\xi,\eta,\zeta)=\xi^{1-2\alpha}\zeta^{1-2\gamma}F_A^{(3)}\left(-\alpha+\beta-\gamma+\frac{5}{2};1-\alpha,\beta,1-\gamma;2-2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\right), \tag{3.24}$$

$$\omega_7(\xi,\eta,\zeta)=\eta^{1-2\beta}\zeta^{1-2\gamma}F_A^{(3)}\left(\alpha-\beta-\gamma+\frac{5}{2};\alpha,1-\beta,1-\gamma;2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\right), \tag{3.25}$$

$$\omega_8(\xi,\eta,\zeta)=\xi^{1-2\alpha}\eta^{1-2\beta}\zeta^{1-2\gamma} \cdot F_A^{(3)}\left(-\alpha-\beta-\gamma+\frac{7}{2};1-\alpha,1-\beta,1-\gamma;2-2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\right). \quad (3.26)$$

Substituting the found solutions (3.19) - (3.26) in to the (3.2), finally we define particular solutions of (3.1)

$$u_1(x,y,z;x_0,y_0,z_0)=\lambda_1\sigma^{-\alpha-\beta-\gamma-\frac{1}{2}}F_A^{(3)}\left(\alpha+\beta+\gamma+\frac{1}{2};\alpha,\beta,\gamma;2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\right), \quad (3.27)$$

$$u_2(x,y,z;x_0,y_0,z_0)=\lambda_2\sigma^{\alpha-\beta-\gamma-\frac{3}{2}}xx_0 \cdot F_A^{(3)}\left(-\alpha+\beta+\gamma+\frac{3}{2};1-\alpha,\beta,\gamma;2-2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\right), \quad (3.28)$$

$$u_3(x,y,z;x_0,y_0,z_0)=\lambda_3\sigma^{-\alpha+\beta-\gamma-\frac{3}{2}}yy_0 \cdot F_A^{(3)}\left(\alpha-\beta+\gamma+\frac{3}{2};\alpha,1-\beta,\gamma;2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\right), \quad (3.29)$$

$$u_4(x,y,z;x_0,y_0,z_0)=\lambda_4\sigma^{-\alpha-\beta+\gamma-\frac{3}{2}}zz_0 \cdot F_A^{(3)}\left(\alpha+\beta-\gamma+\frac{3}{2};\alpha,\beta,1-\gamma;2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\right), \quad (3.30)$$

$$u_5(x,y,z;x_0,y_0,z_0)=\lambda_5\sigma^{\alpha+\beta-\gamma-\frac{5}{2}}xx_0yy_0 \cdot F_A^{(3)}\left(-\alpha-\beta+\gamma+\frac{5}{2};1-\alpha,1-\beta,\gamma;2-2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\right), \quad (3.31)$$

$$u_6(x,y,z;x_0,y_0,z_0)=\lambda_6\sigma^{\alpha-\beta+\gamma-\frac{5}{2}}xx_0zz_0 \cdot F_A^{(3)}\left(-\alpha+\beta-\gamma+\frac{5}{2};1-\alpha,\beta,1-\gamma;2-2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\right), \quad (3.32)$$

$$u_7(x,y,z;x_0,y_0,z_0)=\lambda_7\sigma^{-\alpha+\beta+\gamma-\frac{5}{2}}yy_0zz_0 \cdot F_A^{(3)}\left(\alpha-\beta-\gamma+\frac{5}{2};\alpha,1-\beta,1-\gamma;2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\right), \quad (3.33)$$

$$u_8(x,y,z;x_0,y_0,z_0)=\lambda_8\sigma^{\alpha+\beta+\gamma-\frac{7}{2}}xyzx_0y_0z_0 \cdot F_A^{(3)}\left(-\alpha-\beta-\gamma+\frac{7}{2};1-\alpha,1-\beta,1-\gamma;2-2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\right), \quad (3.34)$$

where $\lambda_i,(i=1,2,...,8)$ are constants and hypergeometric function of Lauricella looks like [5]

$$F_A^{(3)}\left(a;b_1,b_2,b_3;c_1,c_2,c_3;x,y,z\right)=\sum_{i,j,l=0}^{\infty}\frac{(a)_{i+j+l}(b_1)_i(b_2)_j(b_3)_l}{(c_1)_i(c_2)_j(c_3)_l\,i!\,j!\,l!}x^i y^j z^l, \tag{3.35}$$

$$F_A^{(3)}\left(a;b_1,b_2,b_3;c_1,c_2,c_3;x,y,z\right)=\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_3)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(c_1-b_1)\Gamma(c_2-b_2)\Gamma(c_3-b_3)}$$
$$\int_0^1\int_0^1\int_0^1 t_1^{b_1-1}t_2^{b_2-1}t_3^{b_3-1}(1-t_1)^{c_1-b_1-1}(1-t_2)^{c_2-b_2-1}(1-t_3)^{c_3-b_3-1}(1-xt_1-yt_2-zt_3)^{-a}\,dt_1dt_2dt_3, \tag{3.36}$$

$$\mathrm{Re}\,c_1>\mathrm{Re}\,b_1>0,\mathrm{Re}\,c_2>\mathrm{Re}\,b_2>0,\mathrm{Re}\,c_3>\mathrm{Re}\,b_3>0.$$

Here, and in what follows, $(\lambda)_\mu=\Gamma(\lambda+\mu)/\Gamma(\lambda)$ denotes the Pochhammer symbol (or the shifted factorial) for all admissible (real or complex) values of $\lambda$ and $\mu$.

## 4. PROPERTIES OF PARTICULAR SOLUTIONS OF GENERALIZED RASSIAS'S EQUATION

We study properties of particular solutions (3.27) - (3.34). It is not difficult to prove that, the following identities

$$\frac{\partial}{\partial x}u_1\bigg|_{x=0}=0,\ \frac{\partial}{\partial y}u_1\bigg|_{y=0}=0,\ \frac{\partial}{\partial z}u_1\bigg|_{z=0}=0,u_2\big|_{x=0}=0,\ \frac{\partial}{\partial y}u_2\bigg|_{y=0}=0,\ \frac{\partial}{\partial z}u_2\bigg|_{z=0}=0, \tag{4.1}$$

$$\frac{\partial}{\partial x}u_3\bigg|_{x=0}=0,\ u_3\big|_{y=0}=0,\ \frac{\partial}{\partial z}u_3\bigg|_{z=0}=0,\frac{\partial}{\partial x}u_4\bigg|_{x=0}=0,\ \frac{\partial}{\partial y}u_4\bigg|_{y=0}=0,\ u_4\big|_{z=0}=0, \tag{4.2}$$

$$u_5\big|_{x=0}=0,\ u_5\big|_{y=0}=0,\ \frac{\partial}{\partial z}u_5\bigg|_{z=0}=0,u_6\big|_{x=0}=0,\ \frac{\partial}{\partial y}u_6\bigg|_{y=0}=0,\ u_6\big|_{z=0}=0, \tag{4.3}$$

$$\frac{\partial}{\partial x}u_7\bigg|_{x=0}=0,\ u_7\big|_{y=0}=0,\ u_7\big|_{z=0}=0,\ u_8\big|_{x=0}=0,\ u_8\big|_{y=0}=0,\ u_8\big|_{z=0}=0, \tag{4.4}$$

are true.

We investigate behavior of particular solutions (3.27) - (3.34) at $\sigma\to 0$. For this aim we use decomposition ([7], p. 117, (14))

$$\begin{aligned}&F_A^{(3)}\left(a;b_1,b_2,b_3;c_1,c_2,c_3;x,y,z\right)\\&=\sum_{l,m,n=0}^{\infty}\frac{(a)_{l+m+n}(b_1)_{l+m}(b_2)_{l+n}(b_3)_{m+n}}{(c_1)_{l+m}(c_2)_{l+n}(c_3)_{m+n}\,l!m!n!}x^{l+m}y^{l+n}z^{m+n}\\&\cdot{}_2F_1\left(a+l+m,b_1+l+m;c_1+l+m;x\right){}_2F_1\left(a+l+m+n,b_2+l+n;c_2+l+n;y\right)\\&\cdot{}_2F_1\left(a+l+m+n,b_3+m+n;c_3+m+n;z\right),\end{aligned} \tag{4.5}$$

where

$$ {}_2F_1(a,b;c;x)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m\, m!}x^m, $$

is a hypergeometric function of Gauss [5, 6]. By virtue of decomposition (4.5), the particular solution (3.27) can be written as

$$ \begin{aligned} u_1(x,y,z;x_0,y_0,z_0) &= \lambda_1\sigma^{-\alpha-\beta-\gamma-\frac{1}{2}} \\ &\cdot\sum_{l,m,n=0}^{\infty}\frac{\left(\alpha+\beta+\gamma+\frac{1}{2}\right)_{l+m+n}(\alpha)_{l+m}(\beta)_{l+n}(\gamma)_{m+n}}{(2\alpha)_{l+m}(2\beta)_{l+n}(2\gamma)_{m+n}\, l!m!n!}\left(1-\frac{\sigma_1}{\sigma}\right)^{l+m}\left(1-\frac{\sigma_2}{\sigma}\right)^{l+n}\left(1-\frac{\sigma_3}{\sigma}\right)^{m+n} \\ &\cdot{}_2F_1\left(\alpha+\beta+\gamma+\frac{1}{2}+l+m,\alpha+l+m;2\alpha+l+m;1-\frac{\sigma_1}{\sigma}\right) \\ &\cdot{}_2F_1\left(\alpha+\beta+\gamma+\frac{1}{2}+l+m+n,\beta+l+n;2\beta+l+n;1-\frac{\sigma_2}{\sigma}\right) \\ &\cdot{}_2F_1\left(\alpha+\beta+\gamma+\frac{1}{2}+l+m+n,\gamma+m+n;2\gamma+m+n;1-\frac{\sigma_3}{\sigma}\right). \end{aligned} \tag{4.6} $$

We use the following formulae for a hypergeometric function of Gauss [6]

$$ {}_2F_1(a,b;c;x)=(1-x)^{-b}\,{}_2F_1\left(c-a,b;c;\frac{x}{x-1}\right). $$

Then equality (4.6) will have the following form

$$ u_1(x,y,z;x_0,y_0,z_0)=\lambda_1\sigma^{-\frac{1}{2}}\sigma_1^{-\alpha}\sigma_2^{-\beta}\sigma_3^{-\gamma}f(\sigma,\sigma_1,\sigma_2,\sigma_3), \tag{4.7} $$

where

$$ \begin{aligned} f(\sigma,\sigma_1,\sigma_2,\sigma_3) &= \sum_{l,m,n=0}^{\infty}\frac{\left(\alpha+\beta+\gamma+\frac{1}{2}\right)_{l+m+n}(\alpha)_{l+m}(\beta)_{l+n}(\gamma)_{m+n}}{(2\alpha)_{l+m}(2\beta)_{l+n}(2\gamma)_{m+n}\, l!m!n!}\left(\frac{\sigma}{\sigma_1}-1\right)^{l+m}\left(\frac{\sigma}{\sigma_2}-1\right)^{l+n}\left(\frac{\sigma}{\sigma_3}-1\right)^{m+n} \\ &\cdot{}_2F_1\left(\alpha-\beta-\gamma-\frac{1}{2},\alpha+l+m;2\alpha+l+m;1-\frac{\sigma}{\sigma_1}\right) \\ &\cdot{}_2F_1\left(\beta-\alpha-\gamma-\frac{1}{2}-m,\beta+l+n;2\beta+l+n;1-\frac{\sigma}{\sigma_2}\right) \\ &\cdot{}_2F_1\left(\gamma-\alpha-\beta-\frac{1}{2}-l,\gamma+m+n;2\gamma+m+n;1-\frac{\sigma}{\sigma_3}\right). \end{aligned} \tag{4.8} $$

By virtue of equality

$$ {}_2F_1(a,b;c;x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c\neq 0,-1,-2,...,\operatorname{Re}(c-a-b)>0, \tag{4.9} $$

we have

$$
{}_2F_1\left(\alpha-\beta-\gamma-\frac{1}{2},\alpha+l+m;2\alpha+l+m;1-\frac{\sigma}{\sigma_1}\right)\Bigg|_{\sigma=0}
=\frac{\Gamma\left(\beta+\gamma+\frac{1}{2}\right)\Gamma(2\alpha)(2\alpha)_{l+m}}{\Gamma(\alpha)\Gamma\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\left(\alpha+\beta+\gamma+\frac{1}{2}\right)_{l+m}},
\tag{4.10}
$$

$$
{}_2F_1\left(\beta-\alpha-\gamma-\frac{1}{2}-m,\beta+l+n;2\beta+l+n;1-\frac{\sigma}{\sigma_2}\right)\Bigg|_{\sigma=0}
=\frac{\Gamma(2\beta)\Gamma\left(\alpha+\gamma+\frac{1}{2}\right)}{\Gamma(\beta)\Gamma\left(\alpha+\beta+\gamma+\frac{1}{2}\right)}\frac{(2\beta)_{l+n}\left(\alpha+\gamma+\frac{1}{2}\right)_m}{\left(\alpha+\beta+\gamma+\frac{1}{2}\right)_{l+m+n}},
\tag{4.11}
$$

$$
{}_2F_1\left(\gamma-\alpha-\beta-\frac{1}{2}-l,\gamma+m+n;2\gamma+m+n;1-\frac{\sigma}{\sigma_3}\right)\Bigg|_{\sigma=0}
=\frac{\Gamma(2\gamma)\Gamma\left(\alpha+\beta+\frac{1}{2}\right)}{\Gamma(\gamma)\Gamma\left(\alpha+\gamma+\beta+\frac{1}{2}\right)}\frac{(2\gamma)_{m+n}\left(\alpha+\beta+\frac{1}{2}\right)_l}{\left(\alpha+\gamma+\beta+\frac{1}{2}\right)_{l+m+n}}.
\tag{4.12}
$$

Substituting (4.10) - (4.12) into the (4.8) we have

$$
f(0,\sigma_1,\sigma_2,\sigma_3)
=\frac{\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\Gamma\left(\alpha+\gamma+\frac{1}{2}\right)\Gamma\left(\beta+\gamma+\frac{1}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma^3\left(\alpha+\beta+\gamma+\frac{1}{2}\right)}
\cdot\sum_{l,m,n=0}^{\infty}\frac{\left(\alpha+\gamma+\frac{1}{2}\right)_m\left(\alpha+\beta+\frac{1}{2}\right)_l(\alpha)_{l+m}(\beta)_{l+n}(\gamma)_{m+n}}{\left(\alpha+\gamma+\beta+\frac{1}{2}\right)_{l+m+n}\left(\alpha+\beta+\gamma+\frac{1}{2}\right)_{l+m}l!m!n!}.
\tag{4.13}
$$

It is easy to calculate, that

$$
\sum_{l,m,n=0}^{\infty}\frac{\left(\alpha+\gamma+\frac{1}{2}\right)_m\left(\alpha+\beta+\frac{1}{2}\right)_l(\alpha)_{l+m}(\beta)_l(\beta+l)_n(\gamma)_m(\gamma+m)_n}{\left(\alpha+\gamma+\beta+\frac{1}{2}\right)_{l+m+n}\left(\alpha+\beta+\gamma+\frac{1}{2}\right)_{l+m}l!m!n!}
=\frac{\Gamma^2\left(\alpha+\gamma+\beta+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\Gamma\left(\alpha+\gamma+\frac{1}{2}\right)\Gamma\left(\beta+\gamma+\frac{1}{2}\right)}.
\tag{4.14}
$$

Substituting (4.14) in to (4.13), we have

$$f\left(0,\sigma_1,\sigma_2,\sigma_3\right)=\frac{\Gamma\left(2\alpha\right)\Gamma\left(2\beta\right)\Gamma\left(2\gamma\right)\sqrt{\pi}}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma\right)\Gamma\left(\alpha+\beta+\gamma+\frac{1}{2}\right)}. \tag{4.15}$$

By virtue of equality (4.15) from (4.7) follows

$$\left|u_1\left(x,y,z;x_0,y_0,z_0\right)\right|\leq\frac{c}{\sigma^{1/2}\sigma_1^{\alpha}\sigma_2^{\beta}\sigma_3^{\gamma}},\ c-const. \tag{4.16}$$

Expression (4.16) shows that, the particular solution $u_1\left(x,y,z;x_0,y_0,z_0\right)$ is converted to infinity with the order $\sigma^{-1/2}$ at $\sigma\to 0$. Similarly it is possible to be convinced, that particular solutions $u_i\left(x,y,z;x_0,y_0,z_0\right),\ i=2,3,...,8$ are also converted to infinity with the order $\sigma^{-1/2}$ at $\sigma\to 0$.

We study behavior of particular solutions (3.27) - (3.34) at $\sigma\to\infty$. As at $\sigma\to\infty$ arguments of the hypergeometric functions in the solution (3.27) - (3.34) are converted to a zero, i.e. $\xi\to 0,\eta\to 0,\zeta\to 0$ particular solutions $u_i\left(x,y,z;x_0,y_0,z_0\right),\ i=1,2,...,8$ are converted to zero of the order $\sigma^{-\alpha-\beta-\gamma-1/2}$.

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# The solution of the Cauchy problem for generalized Euler-Poisson-Darboux equation

**Anvar Hasanov**

Institute of Mathematics, Uzbek Academy of Sciences,
29 F. Hodjaev Street, Tashkent 700125, Uzbekistan
E-mail: mathinst@uzsci.net, anvarhasanov@yahoo.com

**Abstract**.

*In this paper in a characteristic triangle Cauchy problem for generalized Euler-Poisson-Darboux equation*

$$L_{\alpha,\beta}(u) \equiv u_{\xi\eta} + \left[\frac{\alpha}{\eta+\xi} + \frac{\beta}{\eta-\xi}\right]u_{\xi} + \left[\frac{\alpha}{\eta+\xi} - \frac{\beta}{\eta-\xi}\right]u_{\eta} + \gamma u = 0.$$

*is considered. Function of Riemann, which expressed by Kummer's function of three variables is constructed in an explicit form. By the method of Riemann for the hyperbolic equations, a solution of the Cauchy problem for generalized Euler-Poisson-Darboux equation expressed in an explicit form.*

**2000 Mathematics Subject Classification.** primary 35Q05, 35L80; secondary 33C65.

**Keywords:** Generalized Euler-Poisson-Darboux equation, degenerating hyperbolic type equations, function of Riemann, a Cauchy problem, confluent hypergeometric functions of Kummer from three variables

## 1. INTRODUCTION

Many problems of gas dynamics can be reduced to boundary value problems for the mixed type degenerating equations. It is known that, the mixed type degenerating equations in a hyperbolic part of the domain reduced to the generalized Euler-Poisson-Darboux equation

$$L_{\alpha,\beta}(u) \equiv u_{\xi\eta} + \left[\frac{\alpha}{\eta+\xi} + \frac{\beta}{\eta-\xi}\right]u_{\xi} + \left[\frac{\alpha}{\eta+\xi} - \frac{\beta}{\eta-\xi}\right]u_{\eta} + \gamma u = 0,\ 0 < 2\alpha, 2\beta < 1, \tag{1.1}$$

where $\alpha, \beta$ and $\gamma$ are constant numbers.

The Riemann function of generalized Euler-Poisson-Darboux equation (1.1) was not found. Hence, the Cauchy problem also not solved. Note, Euler-Poisson-Darboux equation

$$L(u) \equiv u_{\xi\eta} + \frac{\beta_1}{\eta-\xi}u_{\xi} - \frac{\alpha_1}{\eta-\xi}u_{\eta} = 0, \alpha_1 > 0, \beta_1 > 0, \alpha_1 + \beta_1 < 1, \tag{1.2}$$

was considered in a work ([1], p. 57) and the Cauchy problem for the equation (1.2) was solved. In the papers [2-4] for the Euler-Poisson-Darboux equation (1.2) non-local boundary value problems in a characteristic triangle were solved. The method of Riemann also is applied to some equations of hyperbolic type [5-15].

In this paper, first we shall introduce two confluent hypergeometric functions from three variables. Further, for confluent hypergeometric functions formulas of an analytic continuation are proved. The function of Riemann for generalized Euler-Poisson-Darboux equation (1.1) by the help of introduced hypergeometric functions is build. Further, in a characteristic triangle with the help of Riemann's function by a classical method we solve a Cauchy problem for generalized Euler-Poisson-Darboux equation (1.1). A solution of the Cauchy problem for generalized Euler-Poisson-Darboux equation (1.1) we shall construct in an explicit form.

## 2. CONSTRUCTIVE PROPERTIES OF GENERALIZED EULER-POISSON-DARBOUX EQUATION

We introduce a new function $v(\xi,\eta)$, supposing

$$u(\xi,\eta)=(\xi-\eta)^{1-2\beta}\,v(\xi,\eta). \tag{2.1}$$

Then the equation (1.1) is reduced to

$$L_{\alpha,1-\beta}v\equiv v_{\xi\eta}+\left(\frac{\alpha}{\eta+\xi}+\frac{1-\beta}{\eta-\xi}\right)v_{\xi}+\left(\frac{\alpha}{\eta+\xi}-\frac{1-\beta}{\eta-\xi}\right)v_{\eta}+\gamma v=0. \tag{2.2}$$

Let's designate through $u_{\alpha,\beta}$ any solution of the equation (1.1). Then by virtue of equality (2.1) we find the first constructive property

$$L_{\alpha,\beta}(u)=(\xi-\eta)^{1-2\beta}\,L_{\alpha,1-\beta}(u). \tag{2.3}$$

Similarly supposing

$$u(\xi,\eta)=(\xi+\eta)^{1-2\alpha}\,v(\xi,\eta), \tag{2.4}$$

we find the second constructive formula

$$L_{\alpha,\beta}(u)=(\xi+\eta)^{1-2\alpha}\,L_{1-\alpha,\beta}(u). \tag{2.5}$$

Note, that constructive formulas (2.3) - (2.5) allow to solve some problems for various values of parameters $\alpha,\beta$ .

## 2. CONFLUENT HYPERGEOMETRIC FUNCTIONS OF KUMMER FROM THREE ARGUMENTS

Consider the system of hypergeometric equations ([16], p. 117)

$$\begin{cases} x(1-x)\omega_{xx}+y\omega_{xy}+z\omega_{xz}+\left[c-(a_1+b_1+1)x\right]\omega_x-a_1b_1\omega=0\\ y(1-y)\omega_{yy}+x\omega_{xy}+z\omega_{yz}+\left[c-(a_2+b_2+1)y\right]\omega_y-a_2b_2\omega=0\\ z(1-z)\omega_{zz}+x\omega_{xz}+y\omega_{yz}+\left[c-(a_3+b_3+1)z\right]\omega_z-a_3b_3\omega=0. \end{cases} \tag{3.1}$$

The solution of the system of hypergeometric equations (3.1) is Lauricella's hypergeometric function ([16], p. 114, (2))

$$F_B^{(3)}\left(a_1,a_2,a_3,b_1,b_2,b_3;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}\,m!n!p!}x^m y^n z^p, \tag{3.2}$$

$$\left(c\neq 0,-1,-2,...,\ s=|x|,r=|y|,t=|z|,s+r+t<1\right).$$

Using identity ([16], p. 124)

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\right)_m\varepsilon^m=\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\right)_n\varepsilon^n=\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\right)_p\varepsilon^p=1, \tag{3.3}$$

where $(\lambda)_\mu=\Gamma(\lambda+\mu)/\Gamma(\lambda)$ is a symbol of Pochhammer (or the *shifted* factorial). From hypergeometric function (3.2), we find the following confluent hypergeometric functions of three variables

$$\begin{aligned}B_1\left(a_1,a_2,a_3,b_1,b_2;c;x,y,z\right)&=\lim_{\varepsilon\to 0}F_B^{(3)}\left(a_1,a_2,a_3,b_1,b_2,\frac{1}{\varepsilon};c;x,y,\varepsilon z\right)=\\&=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_m(b_2)_n}{(c)_{m+n+p}\,m!n!p!}x^m y^n z^p,\end{aligned} \tag{3.4}$$

$$\begin{aligned}B_2\left(a_1,a_2,b_1,b_2;c;x,y,z\right)&=\lim_{\varepsilon\to 0}F_B^{(3)}\left(a_1,a_2,\frac{1}{\varepsilon},b_1,b_2,\frac{1}{\varepsilon};c;x,y,\varepsilon^2 z\right)=\\&=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n+p}\,m!n!p!}x^m y^n z^p.\end{aligned} \tag{3.5}$$

The found hypergeometric functions $B_1$, $B_2$ accordingly satisfy to the following systems of hypergeometric equations

$$\begin{cases}x(1-x)\omega_{xx}+y\omega_{xy}+z\omega_{xz}+\left[c-(a_1+b_1+1)x\right]\omega_x-a_1b_1\omega=0\\ y(1-y)\omega_{yy}+x\omega_{xy}+z\omega_{yz}+\left[c-(a_2+b_2+1)y\right]\omega_y-a_2b_2\omega=0\\ z\omega_{zz}+x\omega_{xz}+y\omega_{yz}+(c-z)\omega_z-a_3\omega=0,\end{cases} \tag{3.6}$$

$$\begin{cases}x(1-x)\omega_{xx}+y\omega_{xy}+z\omega_{xz}+\left[c-(a_1+b_1+1)x\right]\omega_x-a_1b_1\omega=0\\ y(1-y)\omega_{yy}+x\omega_{xy}+z\omega_{yz}+\left[c-(a_2+b_2+1)y\right]\omega_y-a_2b_2\omega=0\\ z\omega_{zz}+x\omega_{xz}+y\omega_{yz}+c\omega_z-\omega=0.\end{cases} \tag{3.7}$$

Confluent hypergeometric functions $B_1$, $B_2$ have accordingly the following integral representations

$$B_1(a_1,a_2,a_3,b_1,b_2;c;x,y,z)=\frac{\Gamma(c)}{\Gamma(b_1)\Gamma(b_2)\Gamma(a_3)\Gamma(c-b_1-b_2-a_3)}$$
$$\cdot\int_0^1\int_0^1\int_0^1 e^{z\xi(1-\eta)\zeta}\xi^{c-b_1-1}\eta^{b_2-1}\zeta^{a_3-1}(1-\xi)^{b_1-1}(1-\eta)^{c-b_1-b_2-1}(1-\zeta)^{c-b_1-b_2-a_3-1} \tag{3.8}$$
$$\cdot[1-x+x\xi]^{-a_1}[1-y\xi\eta]^{-a_2}\,d\xi d\eta d\zeta,$$
$$\operatorname{Re}b_1>0,\operatorname{Re}b_2>0,\operatorname{Re}a_3>0,\operatorname{Re}c>b_1+b_2+a_3>0,$$
$$B_2(a_1,a_2,b_1,b_2;c;x,y,z)=\frac{\Gamma(c)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c-b_1-b_2)}$$
$$\cdot\int_0^1\int_0^1 \xi^{c-b_1-1}\eta^{b_2-1}(1-\xi)^{b_1-1}(1-\eta)^{c-b_1-b_2-1}[1-x+x\xi]^{-a_1}[1-y\xi\eta]^{-a_2} \tag{3.9}$$
$$\cdot\,{}_0F_1[c-b_1-b_2;z\xi(1-\eta)]\,d\xi d\eta,$$
$$\operatorname{Re}b_1>0,\operatorname{Re}b_2>0,\operatorname{Re}c>b_1+b_2>0.$$

Integral representations (3.8) - (3.9) in the case of $z=0$ coincide with earlier known integral ([16], p. 35).

## 3. THE FORMULA OF ANALYTICAL CONTINUATION OF HYPERGEOMETRIC FUNCTION $B_2$

At solving of the Cauchy problem for generalized Euler-Poisson-Darboux equation, we use the analytical continuation formula of confluent hypergeometric function $B_2$. The following formula of analytic continuation is true

$$B_2(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y,z)$$
$$=\frac{\Gamma(\gamma)\Gamma(\beta_2-\alpha_2)}{\Gamma(\beta_2)\Gamma(\gamma-\alpha_2)}(-y)^{-\alpha_2}H_2^{(3)}\left(1-\gamma+\alpha_2;\alpha_1,\alpha_2,\beta_1;1-\beta_2+\alpha_2;\frac{1}{y},-x,-z\right)+ \tag{4.1}$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha_2-\beta_2)}{\Gamma(\alpha_2)\Gamma(\gamma-\beta_2)}(-y)^{-\beta_2}H_2^{(3)}\left(1-\gamma+\beta_2;\alpha_1,\beta_2,\beta_1;1-\alpha_2+\beta_2;\frac{1}{y},-x,-z\right),$$

where [17]

$$H_2^{(3)}(a;b_1,b_2,b_3;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m-n-p}(b_1)_n(b_2)_m(b_3)_n}{(c)_m\,m!n!p!}x^m y^n z^p. \tag{4.2}$$

If $z=0$, then the formula of analytic continuation of confluent hypergeometric function $B_2(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y,z)$ (4.1), coincides with the known formula analytical continuation for hypergeometric Appell's function $F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y)$ ([19], p. 709). We prove the formula of analytic continuation (4.1). For this we use

$$B_2(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y,z)=\sum_{m,p=0}^{\infty}\frac{(\alpha_1)_m(\beta_1)_m}{(\gamma)_{m+p}\,m!p!}x^m z^p F(\alpha_2,\beta_2;\gamma+m+p;y). \tag{4.3}$$

Using the formula analytical continuation for hypergeometric function of Gauss [18]

$$F(a,b;c;y)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-y)^{-a}F\left(a,1-c+a;1-b+a;\frac{1}{y}\right)+ \\ +\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-y)^{-b}F\left(b,1-c+b;1-a+b;\frac{1}{y}\right), \tag{4.4}$$

from expression (4.3) we get

$$\begin{aligned} &B_2(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y,z) \\ &=\frac{\Gamma(\gamma)\Gamma(\beta_2-\alpha_2)}{\Gamma(\beta_2)\Gamma(\gamma-\alpha_2)}(-y)^{-\alpha_2} \\ &\cdot\sum_{m,p=0}^{\infty}\frac{(\alpha_1)_m(\beta_1)_m}{(\gamma-\alpha_2)_{m+p}\,m!\,p!}x^m z^p F\left(\alpha_2,1-\gamma-m-p+\alpha_2;1-\beta_2+\alpha_2;\frac{1}{y}\right)+ \\ &+\frac{\Gamma(\gamma)\Gamma(\alpha_2-\beta_2)}{\Gamma(\alpha_2)\Gamma(\gamma-\beta_2)}(-y)^{-\beta_2} \\ &\cdot\sum_{m,p=0}^{\infty}\frac{(\alpha_1)_m(\beta_1)_m}{(\gamma-\beta_2)_{m+p}\,m!\,p!}x^m z^p F\left(\beta_2,1-\gamma-m-p+\beta_2;1-\alpha_2+\beta_2;\frac{1}{y}\right). \end{aligned} \tag{4.5}$$

Decomposing in a series hypergeometric function of Gauss, we receive

$$\begin{aligned} B_2(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y,z)&=\frac{\Gamma(\gamma)\Gamma(\beta_2-\alpha_2)}{\Gamma(\beta_2)\Gamma(\gamma-\alpha_2)}(-y)^{-\alpha_2} \\ &\cdot\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_m}{(\gamma-\alpha_2)_{m+p}\,m!\,n!\,p!}x^m\left(\frac{1}{y}\right)^n z^p\frac{(1-\gamma-m-p+\alpha_2)_n}{(1-\beta_2+\alpha_2)_n} \\ +\frac{\Gamma(\gamma)\Gamma(\alpha_2-\beta_2)}{\Gamma(\alpha_2)\Gamma(\gamma-\beta_2)}(-y)^{-\beta_2}& \\ &\cdot\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_m(\beta_2)_n(\beta_1)_m}{(\gamma-\beta_2)_{m+p}\,m!\,n!\,p!}x^m\left(\frac{1}{y}\right)^n z^p\frac{(1-\gamma-m-p+\beta_2)_n}{(1-\alpha_2+\beta_2)_n}. \end{aligned} \tag{4.6}$$

By virtue of identity

$$(1-\gamma-m-p+\alpha_2)_n=(\gamma-\alpha_2)_{m+p}(-1)^{m+p}(1-\gamma+\alpha_2)_{-m+n-p},$$

$$(1-\gamma-m-p+\beta_2)_n=(\gamma-\beta_2)_{m+p}(-1)^{m+p}(1-\gamma+\beta_2)_{-m+n-p},$$

and changing the order of summation, from (4.6) we have

$$\begin{aligned} B_2(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y,z)&=\frac{\Gamma(\gamma)\Gamma(\beta_2-\alpha_2)}{\Gamma(\beta_2)\Gamma(\gamma-\alpha_2)}(-y)^{-\alpha_2} \\ &\cdot\sum_{n,m,p=0}^{\infty}\frac{(1-\gamma+\alpha_2)_{m-n-p}(\alpha_1)_n(\alpha_2)_m(\beta_1)_n}{(1-\beta_2+\alpha_2)_m\,n!\,m!\,p!}\left(\frac{1}{y}\right)^m(-x)^n(-z)^p \\ +\frac{\Gamma(\gamma)\Gamma(\alpha_2-\beta_2)}{\Gamma(\alpha_2)\Gamma(\gamma-\beta_2)}(-y)^{-\beta_2}& \\ &\cdot\sum_{n,m,p=0}^{\infty}\frac{(1-\gamma+\beta_2)_{m-n-p}(\alpha_1)_n(\beta_2)_m(\beta_1)_n}{(1-\alpha_2+\beta_2)_m\,n!\,m!\,p!}\left(\frac{1}{y}\right)^m(-x)^n(-z)^p \end{aligned} \tag{4.7}$$

By virtue of definition (4.2) of hypergeometric functions $H_2^{(3)}(a;b_1,b_2,b_3;c;x,y,z)$, from identity (4.7) we shall finally get the formula of an analytic continuation (4.1). Note, in paper [20] expansions of Lauricella's hypergeometric functions from many variables were found.

## 5. THE STATEMENT OF THE CAUCHY PROBLEM

Consider the generalized Euler-Poisson-Darboux equation (1.1) in a characteristic triangle $\Delta$. The triangle is limited by straight lines $\xi=0,\ 0\le\eta\le1$; $\eta=1,\ 0\le\xi\le\eta$; $\eta=\xi$ with tops in points $O(0,0)$, $A(0,1)$, $B(1,1)$.

**Definition.** A regular solution of the equation (1.1) in the domain $\Delta$ we call a function $u(\xi,\eta)$, which is continuous in the closed domain $\Delta$ and has continuous derivatives of the second order in the domain $\Delta$ and satisfying the generalized Euler-Poisson-Darboux equation (1.1).

**Cauchy problem.** Find a regular solution of the equation (1.1) in the domain $\Delta$, satisfying initial conditions

$$\lim_{\eta-\xi\to0} u(\xi,\eta)=\tau(\xi),\quad 0\le\xi\le1, \tag{5.1}$$

$$\lim_{\eta-\xi\to0} (\eta-\xi)^{2\beta}\left(u_\eta-u_\xi\right)=\nu(\xi),\quad 0<\xi<1, \tag{5.2}$$

where $\tau(\xi)\in C(\bar{J})\cap C^2(J),\nu(\xi)\in C(J)\cap C^2(J)$ are given functions, $J=(0,1)$ is an interval of an axis $\eta=0$.

In the theory of degenerating equations of hyperbolic type, the main role is played the function of Riemann $R(\xi,\eta;\xi_0,\eta_0)$, which for the equation (1.1) is defined as follows:

1. $R(\xi,\eta;\xi_0,\eta_0)$ is a solution of the conjugate equation on variables $(\xi,\eta)$

$$M_{\alpha,\beta}(R)\equiv R_{\xi\eta}-\frac{\partial}{\partial\xi}\left\{\left[\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right]R\right\}-\frac{\partial}{\partial\eta}\left\{\left[\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right]R\right\}+\gamma R=0. \tag{5.3}$$

2. The function $R(\xi,\eta;\xi_0,\eta_0)$ on characteristics $\xi=\xi_0$ and $\eta=\eta_0$ accordingly accept values

$$R(\xi_0,\eta;\xi_0,\eta_0)=\left(\frac{\eta+\xi_0}{\eta_0+\xi_0}\right)^{\alpha}\left(\frac{\eta-\xi_0}{\eta_0-\xi_0}\right)^{\beta}, \tag{5.4}$$

$$R(\xi,\eta_0;\xi_0,\eta_0)=\left(\frac{\eta_0+\xi}{\eta_0+\xi_0}\right)^{\alpha}\left(\frac{\eta_0-\xi}{\eta_0-\xi_0}\right)^{\beta}, \tag{5.5}$$

where $\eta\le\eta_0$, $\xi_0\le\xi$, $\xi\le\eta$.

3. The function $R(\xi,\eta;\xi_0,\eta_0)$ is a solution of equation Volterra of the second kind

$$R(\xi,\eta;\xi_0,\eta_0)-\int_{\xi_0}^{\xi}\left[\frac{\alpha}{\eta+t}-\frac{\beta}{\eta-t}\right]R(t,\eta;\xi_0,\eta_0)\,dt$$
$$-\int_{\eta_0}^{\eta}\left[\frac{\alpha}{t+\xi}+\frac{\beta}{t-\xi}\right]R(\xi,t;\xi_0,\eta_0)\,dt+\gamma\int_{\xi_0}^{\xi}\int_{\eta_0}^{\eta}R(t,y;\xi_0,\eta_0)\,dtdy=1. \quad (5.6)$$

Let's construct the function of Riemann of generalized Euler-Poisson-Darboux equation.

## 6. THE FUNCTION OF RIEMANN GENERALIZED EULER-POISSON-DARBOUX EQUATION

The solution of the conjugate equation (5.3) we shall search in the form of

$$R(\xi,\eta;\xi_0,\eta_0)=P\omega(\sigma_1,\sigma_2,\sigma_3), \quad (6.1)$$

where

$$P=\left(\frac{\eta+\xi}{\eta_0+\xi_0}\right)^{\alpha}\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta}, \quad (6.2)$$

$$\sigma_1=-\frac{(\xi-\xi_0)(\eta-\eta_0)}{(\eta+\xi)(\eta_0+\xi_0)},\ \ \sigma_2=\frac{(\xi-\xi_0)(\eta-\eta_0)}{(\eta-\xi)(\eta_0-\xi_0)},\ \ \sigma_3=-\gamma(\xi-\xi_0)(\eta-\eta_0). \quad (6.3)$$

Calculating derivative of expression (6.1) and substituting in to the conjugate equation (5.3), we find

$$A_1\omega_{\sigma_1\sigma_1}+A_2\omega_{\sigma_2\sigma_2}+A_3\omega_{\sigma_3\sigma_3}+B_1\omega_{\sigma_1\sigma_2}+B_2\omega_{\sigma_1\sigma_3}+B_3\omega_{\sigma_2\sigma_3}$$
$$+C_1\omega_{\sigma_1}+C_2\omega_{\sigma_2}+C_3\omega_{\sigma_3}+D\omega=0, \quad (6.4)$$

where

$$A_1=P\sigma_{1\xi}\sigma_{1\eta},\ A_2=P\sigma_{2\xi}\sigma_{2\eta},\ A_3=P\sigma_{3\eta}\sigma_{3\xi},$$

$$B_1=P\left(\sigma_{1\xi}\sigma_{2\eta}+\sigma_{1\eta}\sigma_{2\xi}\right), B_2=P\left(\sigma_{1\xi}\sigma_{3\eta}+\sigma_{1\eta}\sigma_{3\xi}\right), B_3=P\left(\sigma_{2\xi}\sigma_{3\eta}+\sigma_{2\eta}\sigma_{3\xi}\right),$$

$$C_1=\left[P_\xi\sigma_{1\eta}+P_\eta\sigma_{1\xi}+P\sigma_{1\xi\eta}-\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right)P\sigma_{1\xi}-\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right)P\sigma_{1\eta}\right], \quad (6.5)$$

$$C_2=\left[P_\xi\sigma_{2\eta}+P_\eta\sigma_{2\xi}+P\sigma_{2\xi\eta}-\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right)P\sigma_{2\xi}-\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right)P\sigma_{2\eta}\right],$$

$$C_3=\left[P_\xi\sigma_{3\eta}+P_\eta\sigma_{3\xi}+P\sigma_{3\xi\eta}-\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right)P\sigma_{3\xi}-\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right)P\sigma_{3\eta}\right],$$

$$D=\left[P_{\xi\eta}-\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right)P_\xi-\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right)P_\eta+\left(\frac{2\alpha}{(\eta+\xi)^2}-\frac{2\beta}{(\eta-\xi)^2}+\gamma\right)P\right].$$

After elementary evaluations from (6.5) we get

$$A_1=-\frac{P}{(\eta+\xi)^2}\sigma_1(1-\sigma_1),\ A_2=P\frac{1}{(\eta-\xi)^2}\sigma_2(1-\sigma_2),\ A_3=-\gamma P\sigma_3,$$

$$B_1=P\left[-\frac{1}{(\eta+\xi)^2}\sigma_2+\frac{1}{(\eta-\xi)^2}\sigma_1\right], B_2=-P\gamma\sigma_1-P\frac{\sigma_3}{(\eta+\xi)^2}, B_3=-\gamma P\sigma_2+P\frac{\sigma_3}{(\eta-\xi)^2},$$

$$C_1=-P\frac{1}{(\eta+\xi)^2}\left[1-(1+\alpha+1-\alpha)\sigma_1\right],\ C_2=\frac{P}{(\eta-\xi)^2}\left[1-(1+\beta+1-\beta)\sigma_2\right],\ C_3=-\gamma P, \tag{6.6}$$

$$D=P\frac{\alpha(1-\alpha)}{(\eta+\xi)^2}+P\frac{\beta(\beta-1)}{(\eta-\xi)^2}+P\gamma.$$

Substituting the received expressions (6.6) in to the equation (6.4), we define system of hypergeometric equation

$$\begin{cases}\sigma_1(1-\sigma_1)\omega_{\sigma_1\sigma_1}+\sigma_2\omega_{\sigma_1\sigma_2}+\sigma_3\omega_{\sigma_1\sigma_3}+\left[1-(1+\alpha+1-\alpha)\sigma_1\right]\omega_{\sigma_1}-\alpha(1-\alpha)\omega=0\\ \sigma_2(1-\sigma_2)\omega_{\sigma_2\sigma_2}+\sigma_1\omega_{\sigma_1\sigma_2}+\sigma_3\omega_{\sigma_2\sigma_3}+\left[1-(1+\beta+1-\beta)\sigma_2\right]\omega_{\sigma_2}-\beta(1-\beta)\omega=0\\ \sigma_3\omega_{\sigma_3\sigma_3}+\sigma_1\omega_{\sigma_1\sigma_3}+\sigma_2\omega_{\sigma_2\sigma_3}+\omega_{\sigma_3}-\omega=0.\end{cases} \tag{6.7}$$

Considering a solution of the system of hypergeometric equations (3.7), we can find a solution of system (6.7)

$$\omega(\sigma_1,\sigma_2,\sigma_3)=B_2(\alpha,\beta,1-\alpha,1-\beta;1;\sigma_1,\sigma_2,\sigma_3). \tag{6.8}$$

Substituting a solution (6.8) in to the representation (6.1), we define function of Riemann for the Cauchy problem

$$R(\xi,\eta;\xi_0,\eta_0)=\left(\frac{\eta+\xi}{\eta_0+\xi_0}\right)^{\alpha}\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta}B_2(\alpha,\beta,1-\alpha,1-\beta;1;\sigma_1,\sigma_2,\sigma_3). \tag{6.9}$$

Further, using function of Riemann (6.9), we shall solve the Cauchy problem for generalized Euler-Poisson-Darboux equation (1.1).

## 7. THE SOLUTION OF THE CAUCHY PROBLEM FOR GENERALIZED EULER-POISSON-DARBOUX EQUATION

We designate through $\Delta_\varepsilon$ the domain, limited by a segment $P_1P_2$ of a straight line $\eta=\xi+\varepsilon$ $(\varepsilon>0)$ and characteristics $PP_1:\xi=\xi_0$, $PP_2:\eta=\eta_0$. The following identity is true

$$2\left[RL_{\alpha,\beta}(u)-uM_{\alpha,\beta}(R)\right]=$$
$$=\frac{\partial}{\partial\eta}\left[Ru_\xi-uR_\xi+\left(\frac{2\alpha}{\eta+\xi}-\frac{2\beta}{\eta-\xi}\right)uR\right]+\frac{\partial}{\partial\xi}\left[Ru_\eta-uR_\eta+\left(\frac{2\alpha}{\eta+\xi}+\frac{2\beta}{\eta-\xi}\right)uR\right]. \tag{7.1}$$

Integrating identity (7.1) on domain $\Delta_\varepsilon$ and applying Green's formula, we have

$$\int_{\partial\Delta_\varepsilon}\left[Ru_\eta - uR_\eta + \left(\frac{2\alpha}{\eta+\xi}+\frac{2\beta}{\eta-\xi}\right)uR\right]d\eta - \left[Ru_\xi - uR_\xi + \left(\frac{2\alpha}{\eta+\xi}-\frac{2\beta}{\eta-\xi}\right)uR\right]d\xi = 0, \qquad (7.2)$$

where $\partial\Delta_\varepsilon = P_1P_2 \cup PP_1 \cup PP_2$ is a boundary of domain $\Delta_\varepsilon$; taking $PP_1 : d\xi = 0$, $PP_2 : d\eta = 0$ and properties of the function of Riemann (6.9) into account , last identity we rewrite in the form of

$$u(\xi_0,\eta_0) = \int_{\xi_0}^{\eta_0-\varepsilon}\left\{\left[\frac{R_\xi - R_\eta}{2} + \frac{2\beta}{\eta-\xi}R\right]u\right\}\Bigg|_{\eta=\xi+\varepsilon} d\xi + \frac{1}{2}\int_{\xi_0}^{\eta_0-\varepsilon}\left[u_\eta - u_\xi\right]R\Big|_{\eta=\xi+\varepsilon} d\xi. \qquad (7.3)$$

Calculate the first integral in a solution (7.3). Using the formula of the analytic continuation (4.1), from function of Riemann (6.9) we define

$$\begin{aligned} R(\xi,\eta;\xi_0,\eta_0) &= \lambda_1 P_1 H_2^{(3)}(\beta;\alpha,\beta,1-\alpha;2\beta;\rho_1,\rho_2,\rho_3) \\ &\quad + \lambda_2 P_2 H_2^{(3)}(1-\beta;\alpha,1-\beta,1-\alpha;2-2\beta;\rho_1,\rho_2,\rho_3), \end{aligned} \qquad (7.4)$$

where

$$\begin{aligned} &P_1 = \frac{(\eta+\xi)^\alpha(\eta-\xi)^{2\beta}}{(\eta_0+\xi_0)^\alpha(\eta_0-\eta)^\beta(\xi-\xi_0)^\beta},\ P_2 = \frac{(\eta+\xi)^\alpha(\eta-\xi)(\eta_0-\xi_0)^{1-2\beta}}{(\eta_0+\xi_0)^\alpha(\eta_0-\eta)^{1-\beta}(\xi-\xi_0)^{1-\beta}}, \\ &\lambda_1 = \frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)},\ \lambda_2 = -\frac{\Gamma(2\beta)}{(1-2\beta)\Gamma^2(\beta)}, \\ &\rho_1 = \frac{(\eta-\xi)(\eta_0-\xi_0)}{(\xi-\xi_0)(\eta-\eta_0)},\ \rho_2 = \frac{(\xi-\xi_0)(\eta-\eta_0)}{(\eta+\xi)(\eta_0+\xi_0)},\ \rho_3 = \gamma(\eta-\eta_0)(\xi-\xi_0). \end{aligned} \qquad (7.5)$$

Let's calculate derivatives of the function (7.4). Then we find

$$\begin{aligned} R_\xi &= \lambda_1 P_{1\xi} H_2^{(3)}(2\beta) + \lambda_1 P_1 H_{2\rho_1}^{(3)}(2\beta)\rho_{1\xi} + \lambda_1 P_1 H_{2\rho_2}^{(3)}(2\beta)\rho_{2\xi} \\ &+ \lambda_1 P_1 H_{2\rho_3}^{(3)}(2\beta)\rho_{3\xi} + \lambda_2 P_{2\xi} H_2^{(3)}(2-2\beta) + \lambda_2 P_2 H_{2\rho_1}^{(3)}(2-2\beta)\rho_{1\xi} \\ &+ \lambda_2 P_2 H_{2\rho_2}^{(3)}(2-2\beta)\rho_{2\xi} + \lambda_2 P_2 H_{2\rho_3}^{(3)}(2-2\beta)\rho_{3\xi}, \end{aligned} \qquad (7.6)$$

$$\begin{aligned} R_\eta &= \lambda_1 P_{1\eta} H_2^{(3)}(2\beta) + \lambda_1 P_1 H_{2\rho_1}^{(3)}(2\beta)\rho_{1\eta} + \lambda_1 P_1 H_{2\rho_2}^{(3)}(2\beta)\rho_{2\eta} \\ &+ \lambda_1 P_1 H_{2\rho_3}^{(3)}(2\beta)\rho_{3\eta} + \lambda_2 P_{2\eta} H_2^{(3)}(2-2\beta) + \lambda_2 P_2 H_{2\rho_1}^{(3)}(2-2\beta)\rho_{1\eta} \\ &+ \lambda_2 P_2 H_{2\rho_2}^{(3)}(2-2\beta)\rho_{2\eta} + \lambda_2 P_2 H_{2\rho_3}^{(3)}(2-2\beta)\rho_{3\eta}, \end{aligned} \qquad (7.7)$$

where

$$H_2^{(3)}(2\beta) = H_2^{(3)}(\beta;\alpha,\beta,1-\alpha;2\beta;\rho_1,\rho_2,\rho_3),$$
$$H_2^{(3)}(2-2\beta) = H_2^{(3)}(1-\beta;\alpha,1-\beta,1-\alpha;2-2\beta;\rho_1,\rho_2,\rho_3).$$

Substituting found derivatives of the function of Riemann (7.6) - (7.7), in to the first integral of the solution (7.3) we get

$$\begin{aligned}&\frac{R_\xi - R_\eta}{2} + \frac{2\beta}{\eta-\xi} R \\ &= \frac{1}{2}\lambda_1 H_2^{(3)}\left(2\beta\right)\left[P_{1\xi} - P_{1\eta}\right] + \frac{1}{2}\lambda_2 P_{2\xi} H_2^{(3)}\left(2-2\beta\right)\left[P_{2\xi} - P_{2\eta}\right] \\ &+ \frac{1}{2}\left[\lambda_1 P_1 H_{2\rho_1}^{(3)}\left(2\beta\right) + \lambda_2 P_2 H_{2\rho_1}^{(3)}\left(2-2\beta\right)\right]\left(\rho_{1\xi} - \rho_{1\eta} + \rho_{2\xi} - \rho_{2\eta} + \rho_{3\xi} - \rho_{3\eta}\right) \\ &+ \lambda_1 P_1 \frac{2\beta}{\eta-\xi} H_2^{(3)}\left(2\beta\right) + \lambda_2 P_2 \frac{2\beta}{\eta-\xi} H_2^{(3)}\left(2-2\beta\right).\end{aligned} \tag{7.8}$$

Considering the formula of derivation

$$\begin{aligned}&\frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k} H_2^{(3)}\left(a; b_1, b_2, b_3; c; x, y, z\right) \\ &= \frac{(a)_{i-j-k} (b_1)_j (b_2)_i (b_3)_j}{(c)_i} H_2^{(3)}\left(a+i-j-k; b_1+j, b_2+i, b_3+j; c+i; x, y, z\right),\end{aligned} \tag{7.9}$$

and also by virtue of equalities

$$P_{1\xi} - P_{1\eta} = -P_1\left(\frac{4\beta}{\eta-\xi} + \frac{\beta}{\xi-\xi_0} + \frac{\beta}{\eta_0-\eta}\right),$$

$$P_{2\xi} - P_{2\eta} = -P_2\left(\frac{2}{\eta-\xi} + \frac{1-\beta}{\xi-\xi_0} + \frac{1-\beta}{\eta_0-\eta}\right),$$

$$\rho_{1\xi} - \rho_{1\eta} = \frac{(\eta_0-\xi_0)(\xi_0-\eta)}{(\eta-\eta_0)(\xi-\xi_0)^2} - \frac{(\eta_0-\xi_0)(\xi-\eta_0)}{(\eta-\eta_0)^2(\xi-\xi_0)},$$

$$\rho_{2\xi} - \rho_{2\eta} = \frac{(\eta-\eta_0)(\eta+\xi_0)}{(\eta_0+\xi_0)(\eta+\xi)^2} - \frac{(\xi-\xi_0)(\xi+\eta_0)}{(\eta_0+\xi_0)(\eta+\xi)^2},$$

$$\rho_{3\xi} - \rho_{3\eta} = \gamma(\eta-\eta_0) - \gamma(\xi-\xi_0) = \gamma\left[\eta-\xi+\xi_0-\eta_0\right],$$

from expression (7.8), follows

$$\begin{aligned}&\frac{R_\xi - R_\eta}{2} + \frac{2\beta}{\eta-\xi} R = -\frac{1}{2}\lambda_1 P_1 H_2^{(3)}\left(2\beta\right)\left(\frac{\beta}{\xi-\xi_0} + \frac{\beta}{\eta_0-\eta}\right) \\ &+ \lambda_2 P_2 H_2^{(3)}\left(2-2\beta\right)\left(\frac{2\beta-1}{\eta-\xi} - \frac{1}{2}\frac{1-\beta}{\xi-\xi_0} - \frac{1}{2}\frac{1-\beta}{\eta_0-\eta}\right) \\ &+ \frac{1}{2}\left[\lambda_1 P_1 H_{2\rho_1}^{(3)}\left(2\beta\right) + \lambda_2 P_2 H_{2\rho_1}^{(3)}\left(2-2\beta\right)\right]\left(\rho_{1\xi} - \rho_{1\eta} + \rho_{2\xi} - \rho_{2\eta} + \rho_{3\xi} - \rho_{3\eta}\right).\end{aligned} \tag{7.10}$$

Identity (7.10) we substitute in to the first integral of the solution (7.3) and we define

$$\begin{aligned}&\lim_{\varepsilon\to 0} \int_{\xi_0}^{\eta_0-\varepsilon} \left\{\left[\frac{v_\xi - v_\eta}{2} + \frac{2\beta}{\eta-\xi} v\right] u\right\}\Bigg|_{\eta=\xi+\varepsilon} d\xi = \lambda_2\left(2\beta-1\right)\frac{(\eta_0-\xi_0)^{1-2\beta}}{(\eta_0+\xi_0)^{\alpha}} \\ &\cdot \int_{\xi_0}^{\eta_0} \frac{(2\xi)^{\alpha}\,\tau(\xi)}{(\eta_0-\xi)^{1-\beta}(\xi-\xi_0)^{1-\beta}} H_2^{(3)}\left(1-\beta; \alpha, 1-\beta, 1-\alpha; 2-2\beta; 0, \rho_2|_{\eta=\xi}, \rho_3|_{\eta=\xi}\right) d\xi.\end{aligned} \tag{7.11}$$

Taking

$$H_2^{(3)}\left(1-\beta;\alpha,1-\beta,1-\alpha;2-2\beta;0,\rho_2,\rho_3\right)=\Xi_2\left(\alpha,1-\alpha;\beta;-\rho_2,-\rho_3\right), \tag{7.12}$$

into account, where confluent hypergeometric function of the Horn [16, 18] $\Xi_2\left(\alpha,1-\alpha;\beta;-\rho_2,-\rho_3\right)$ has a form

$$\Xi_2\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m (b)_m}{(c)_{m+n}\, m!n!}x^m y^n. \tag{7.13}$$

Then the relation (7.11) will become

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\xi_0}^{\eta_0-\varepsilon}\left.\left\{\left[\frac{v_\xi-v_\eta}{2}+\frac{2\beta}{\eta-\xi}v\right]u\right\}\right|_{\eta=\xi+\varepsilon}d\xi\\ &=\lambda_2\left(2\beta-1\right)\frac{\left(\eta_0-\xi_0\right)^{1-2\beta}}{\left(\eta_0+\xi_0\right)^{\alpha}}\int_{\xi_0}^{\eta_0}\frac{\left(2\xi\right)^{\alpha}\tau\left(\xi\right)}{\left(\eta_0-\xi\right)^{1-\beta}\left(\xi-\xi_0\right)^{1-\beta}}\Xi_2\left(\alpha,1-\alpha;\beta;\vartheta_1,\vartheta_2\right)d\xi,\end{aligned} \tag{7.14}$$

where

$$\vartheta_1=\frac{\left(\eta_0-\xi\right)\left(\xi-\xi_0\right)}{2\xi\left(\eta_0+\xi_0\right)},\ \vartheta_2=\gamma\left(\eta_0-\xi\right)\left(\xi-\xi_0\right). \tag{7.15}$$

Further, we calculate the second integral in the formula (7.36). For this purpose in the expression (6.9), we shall select hypergeometric function on argument $\sigma_2$.

$$\begin{aligned}&R\left(\xi,\eta;\xi_0,\eta_0\right)=\\ &=\left(\frac{\eta+\xi}{\eta_0+\xi_0}\right)^{\alpha}\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta}\sum_{m,p=0}^{\infty}\frac{(\alpha)_m(1-\alpha)_m}{(1)_{m+p}\,m!\,p!}F\left(\beta,1-\beta;1+m+p;\sigma_2\right)\sigma_1^m\sigma_3^p.\end{aligned} \tag{7.16}$$

We use the following formula [18]

$$F\left(a,b;c;z\right)=\left(1-z\right)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right), \tag{7.17}$$

and from expression (7.16), we find

$$\begin{aligned}&R\left(\xi,\eta;\xi_0,\eta_0\right)=\left(\frac{\eta+\xi}{\eta_0+\xi_0}\right)^{\alpha}\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta}\left(1-\sigma_2\right)^{-\beta}\\ &\cdot\sum_{m,p=0}^{\infty}\frac{(\alpha)_m(1-\alpha)_m}{(1)_{m+p}\,m!\,p!}F\left(\beta,\beta+m+p;1+m+p;\frac{\sigma_2}{\sigma_2-1}\right)\sigma_1^m\sigma_3^p.\end{aligned} \tag{7.18}$$

Considering equality

$$1-\sigma_2=\frac{\left(\eta-\xi_0\right)\left(\eta_0-\xi\right)}{\left(\eta-\xi\right)\left(\eta_0-\xi_0\right)},\ \sigma_2^*=\frac{\sigma_2}{\sigma_2-1}=-\frac{\left(\xi-\xi_0\right)\left(\eta-\eta_0\right)}{\left(\eta-\xi_0\right)\left(\eta_0-\xi\right)}, \tag{7.19}$$

we have

$$R(\xi,\eta;\xi_0,\eta_0)=\frac{(\eta+\xi)^{\alpha}(\eta-\xi)^{2\beta}}{(\eta_0+\xi_0)^{\alpha}(\eta-\xi_0)^{\beta}(\eta_0-\xi)^{\beta}}$$
$$\cdot\sum_{m,p=0}^{\infty}\frac{(\alpha)_m(1-\alpha)_m}{(1)_{m+p}\,m!\,p!}F\left(\beta,\beta+m+p;1+m+p;\sigma_2^*\right)\sigma_1^m\sigma_3^p. \tag{7.20}$$

Now by the help of (7.21) we calculate expression under the second integral in the formula (7.3). Really, taking (7.20) into account, we define

$$\lim_{\varepsilon\to 0}\left[u_\eta-u_\xi\right]R\Big|_{\eta=\xi+\varepsilon}=\nu(\xi)\lim_{\varepsilon\to 0}\frac{(\eta+\xi)^{\alpha}}{(\eta_0+\xi_0)^{\alpha}(\eta-\xi_0)^{\beta}(\eta_0-\xi)^{\beta}}$$
$$\cdot\sum_{m,p=0}^{\infty}\frac{(\alpha)_m(1-\alpha)_m}{(1)_{m+p}\,m!\,p!}F\left(\beta,\beta+m+p;1+m+p;\sigma_2^*\right)\sigma_1^m\sigma_3^p\Big|_{\eta=\xi+\varepsilon}. \tag{7.21}$$

Using the following limits

$$\sigma_1^*=\lim_{\varepsilon\to 0}\sigma_1\big|_{\eta=\xi+\varepsilon}=-\frac{(\xi-\xi_0)(\xi-\eta_0)}{(2\xi)(\eta_0+\xi_0)}=\frac{(\eta_0-\xi)(\xi-\xi_0)}{2\xi(\eta_0+\xi_0)}, \tag{7.22}$$
$$\lim_{\varepsilon\to 0}\sigma_2^*\Big|_{\eta=\xi+\varepsilon}=1,\quad \sigma_3^*=\lim_{\varepsilon\to 0}\sigma_3\big|_{\eta=\xi+\varepsilon}=\gamma(\eta_0-\xi)(\xi-\xi_0),$$

and also value of hypergeometric function of Gauss in a point $\sigma_2^*=1$ [18]

$$F(\beta,\beta+m+p;1+m+p;1)=\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\frac{(1)_{m+p}}{(1-\beta)_{m+p}},$$

from (7.22), we define

$$\lim_{\varepsilon\to 0}\left[u_\eta-u_\xi\right]R\Big|_{\eta=\xi+\varepsilon}$$
$$=\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\nu(\xi)\frac{(2\xi)^{\alpha}}{(\eta_0+\xi_0)^{\alpha}(\eta_0-\xi)^{\beta}(\xi-\xi_0)^{\beta}}\Xi_2\left(\alpha,1-\alpha;1-\beta;\vartheta_1,\vartheta_2\right), \tag{7.23}$$

where $\vartheta_1,\vartheta_2$ are defined by equality (7.15). Thus, the second integral in the formula (7.3) has a form

$$\frac{1}{2}\lim_{\varepsilon\to 0}\int_{\xi_0}^{\eta_0-\varepsilon}\left[u_\eta-u_\xi\right]R\Big|_{\eta=\xi+\varepsilon}d\xi$$
$$=\frac{1}{2}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\int_{\xi_0}^{\eta_0}\frac{(2\xi)^{\alpha}\nu(\xi)}{(\eta_0+\xi_0)^{\alpha}(\eta_0-\xi)^{\beta}(\xi-\xi_0)^{\beta}}\Xi_2\left(\alpha,1-\alpha;1-\beta;\vartheta_1,\vartheta_2\right)d\xi. \tag{7.24}$$

Substituting equalities (7.14) and (7.24) in to the formula (7.3), finally we find a solution of the Cauchy problem of generalized Euler-Poisson-Darboux equation

$$u\left(\xi_0,\eta_0\right)=k_1\frac{\left(\eta_0-\xi_0\right)^{1-2\beta}}{\left(\eta_0+\xi_0\right)^{\alpha}}\int_{\xi_0}^{\eta_0}\frac{\xi^{\alpha}\tau\left(\xi\right)}{\left(\eta_0-\xi\right)^{1-\beta}\left(\xi-\xi_0\right)^{1-\beta}}\Xi_2\left(\alpha,1-\alpha;\beta;\vartheta_1,\vartheta_2\right)d\xi$$
$$+k_2\frac{1}{\left(\eta_0+\xi_0\right)^{\alpha}}\int_{\xi_0}^{\eta_0}\frac{\xi^{\alpha}\nu\left(\xi\right)}{\left(\eta_0-\xi\right)^{\beta}\left(\xi-\xi_0\right)^{\beta}}\Xi_2\left(\alpha,1-\alpha;1-\beta;\vartheta_1,\vartheta_2\right)d\xi, \qquad (7.25)$$

where $\vartheta_1,\vartheta_2$ are defined by equality (7.15) and

$$k_1=2^{\alpha}\frac{\Gamma\left(2\beta\right)}{\Gamma^2\left(\beta\right)},k_2=2^{\alpha-1}\frac{\Gamma\left(1-2\beta\right)}{\Gamma^2\left(1-\beta\right)}. \qquad (7.26)$$

It is not difficult to be convinced that, solutions of the Cauchy problem (7.26) satisfy to conditions of the Cauchy problem (5.1) - (5.2), and generalized Euler-Poisson-Darboux equation.

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# Uniqueness of the Solution of One Non-Local Boundary Value Problem for the Mixed Parabolic Type Equation

**E. T. Karimov**

Insitute of Mathematics, Uzbek Academy of Sciences
F. Khodzhaev str. 29, Tashkent, 700125, Uzbekistan
E-mail: erkinjon@gmail.com

**ABSTRACT**

*In the present work uniqueness of solution of boundary value problem with non-local conditions for mixed parabolic type equation is proven by the method of energy integrals. Moreover, using Fourier method eigenvalues and eigenfunctions of the considered problem are found.*

**Keywords and Phrases:** Mixed Parabolic type equation; non-local boundary value problem; eigenvalue and eigenfunction of problem.

**Mathematics Subject Classification:** 35M10; 35P05

## 1 INTRODUCTION

It is known that the theory of non-local problems is actual by itself as a division of general theory of boundary-value problems for the partial differential equations, and as a division of mathematics, which has numerous applications to the mechanics, physics, biology and to other material sciences.

Note the work [1], where given main results on linear second order partial differential equations of parabolic type and the works [2-5], where studied several boundary problems for mixed type equations. In the book of J. M. Rassias [6] one can find detailed explanations of methods using at solving boundary value problems for mixed type equations. Especially, note works of A.L. Kerefov [7], E. T. Karimov [8], A. S. Berdyshev and E. T. Karimov [9], and works of N. N. Shopolov [10] devoted to studying non-local boundary value problems for mixed parabolic and mixed parabolic-hyperbolic type equations, which results closely related to the present work.

## 2 FORMULATION OF THE PROBLEM

We consider the equation

$$0=\begin{cases}u_{xx}+u_y-\lambda_1 u, & x<0\\ u_{xx}-u_y-\lambda_2 u, & x>0\end{cases} \tag{2.1}$$

in the domain $D=D_1\cup D_2$, in which $D_1=\{(x,y):-1<x<0,\ 0\le y\le 1\}$, $D_2=\{(x,y):0<x<1,\ 0\le y\le 1\}$, $\lambda_1,\ \lambda_2$ are given complex numbers.

**The Problem C.** *Find a function* $u(x,y)$*, which satisfies the following conditions:*

i) $u(x,y)\in C\left(\overline{D}\right)\cap C^{2,1}_{x,y}\left(D_1\cup D_2\right)$; (2.2)

ii) $u_x(x,y)\in C\left(D\cup I_1\cup I_3\cup I_3\right)$; (2.3)

iii) satisfies equation (2.1) in domain $D_1\cup D_2$;

iv) $u(-1,y)=\varphi_1(y),\quad u(1,y)=\varphi_2(y),\quad 0\le y\le 1$; (2.4)

v) $u(x,0)-u(x,1)=\psi_1(x),\ -1\le x\le 0,$ (2.5)

$u(x,0)+u(x,1)=\psi_2(x),\ 0\le x\le 1,$ (2.6)

*where*

$I_1=\{(x,y):x=-1,0\le y\le 1\}$, $I_2=\{(x,y):x=0,0\le y\le 1\}$, $I_3=\{(x,y):x=1,0\le y\le 1\}$, $\varphi_i(y),\ \psi_i(x)\ (i=1,2)$ are given real valued functions, moreover $\psi_1(0)=\psi_2(0)$, $\varphi_1(0)-\varphi_1(1)=\psi_1(-1),\ \varphi_2(0)+\varphi_2(1)=\psi_2(1)$.

## 3 UNIQUENESS OF THE SOLUTION

Let the Problem C has two solutions $u_1$ and $u_2$. We designate through $U=u_1-u_2$ and obtain the following homogeneous problem C:

**Homogeneous problem C.** Find a solution of the equation

$$0=\begin{cases}U_{xx}+U_y-\lambda_1 U, & x<0\\ U_{xx}-U_y-\lambda_2 U, & x>0,\end{cases} \tag{3.1}$$

which satisfies conditions (2.2), (2.3), and the following conditions:

$U(-1,y)=0,\quad U(1,y)=0,\quad 0\le y\le 1$; (3.2)

$U(x,0)-U(x,1)=0,\ -1\le x\le 0,$ (3.3)

$U(x,0)+U(x,1)=0,\ 0\le x\le 1.$ (3.4)

**Theorem.** *If* $\operatorname{Re}(\lambda_1+\lambda_2)\geq 0$ *is valid, then homogeneous problem C has trivial solution.*

Proof:

We prove this theorem by the method of energy integrals [6]. For this at $x<0$ we multiply equation (2.1) to $\overline{U(x,y)}$, which is complex-conjugate function of $U(x,y)$:

$$\overline{U}\left(U_{xx}+U_y-\lambda_1 U\right)\equiv 0,$$

and integrate along the domain $D_1^{\varepsilon}=\{(x,y):-1+\varepsilon<x\leq-\varepsilon<0,\ 0\leq y\leq 1-\varepsilon\}$:

$$\iint_{D_1^{\varepsilon}}\left(\left|U_x\right|^2-\left(\overline{U}U_x\right)_x-\frac{1}{2}\left(|U|^2\right)_y+\lambda_1|U|^2\right)dxdy\equiv 0.$$

Using Green's formula we have [6]

$$\iint_{D_1^{\varepsilon}}\left(\left|U_x\right|^2+\lambda_1|U|^2\right)dxdy-\int_0^{1-\varepsilon}\left[\overline{U(-\varepsilon,y)}U_x(-\varepsilon,y)-\overline{U(-1+\varepsilon,y)}U_x(-1+\varepsilon,y)\right]dy-$$
$$-\frac{1}{2}\int_{-1+\varepsilon}^{-\varepsilon}\left[|U(x,1-\varepsilon)|^2-|U(x,0)|^2\right]dx\equiv 0. \tag{3.5}$$

Similarly at $x>0$ we get the following equality:

$$\iint_{D_2^{\varepsilon}}\left(\left|U_x\right|^2+\lambda_2|U|^2\right)dxdy-\int_{\varepsilon}^{1}\left[\overline{U(1-\varepsilon,y)}U_x(1-\varepsilon,y)-\overline{U(\varepsilon,y)}U_x(\varepsilon,y)\right]dy-$$
$$-\frac{1}{2}\int_{\varepsilon}^{1-\varepsilon}\left[|U(x,1)|^2-|U(x,\varepsilon)|^2\right]dx\equiv 0, \tag{3.6}$$

in which $D_2^{\varepsilon}=\{(x,y):0<x\leq 1-\varepsilon,\ \varepsilon\leq y\leq 1\}$.

We pass to the limit when $\varepsilon\to 0$ and separate the real parts of (3.5), (3.6). After taking the following equality

$$\operatorname{Re}\left(\overline{U(-1,y)}U_x(-1,y)\right)=\operatorname{Re}\left(U(-1,y)\overline{U_x(-1,y)}\right)$$

and conditions (3.2) – (3.4) into account we obtain

$$\iint_{D}\left(\left|U_x\right|^2+\operatorname{Re}\left(\lambda_1+\lambda_2\right)|U|^2\right)dxdy+\int_0^1|U(x,0)|^2\,dx=0. \tag{3.7}$$

Further assume that $\operatorname{Re}(\lambda_1+\lambda_2)\geq 0$. Finally, taking condition (3.2) into account we get $u(x,y)=0$ in the domain $\overline{D}$.

Theorem is proved.

It is easy to conclude that if homogeneous problem C has trivial solution, then problem C has unique solution.

## 4 EIGENVALUES AND EIGENFUNCTIONS OF THE PROBLEM C

**Definition.** *We call values of* $\lambda_i\,(i=1,2)$ *as an* ***eigenvalues of the problem C****, if at these values of* $\lambda_i\,(i=1,2)$ *homogeneous problem C has non-trivial solution. These non-trivial solutions we call as* ***eigenfunctions of the problem C****.*

**Corollary 1.** *The Problem C can have eigenvalues only outside of the domain* $\Delta=\left\{\lambda_i : \operatorname{Re}\lambda_i \geq 0\right\} \quad (i=1,2)$.

The solution of homogeneous problem C can be written as

$$U(x,y)=X(x)Y(y). \qquad (4.1)$$

Substituting (4.1) into the equation (3.1) at $x<0$ and supposing $X(x)\neq 0,\ Y(y)\neq 0$, after some transformation we obtain

$$\frac{X'(x)}{X(x)}-\lambda_1=\frac{Y'(y)}{Y(y)}=\mu_1, \qquad (4.2)$$

where $\mu_1$ is any constant. From (4.2) we get

$$\left[\begin{array}{l} X''(x)-\left(\lambda_1+\mu_1\right)X(x)=0 \\ Y'(y)+\mu_1 Y(y)=0. \end{array}\right. \qquad (4.3)$$

The condition (3.2) will pass to $X(-1)=0,\quad X(0)=0$. And condition (3.3) will have the form of $Y(0)=-Y(-1)$. After the solving the ordinary partial differential equations (4.3) by the variation method at $\lambda_1^{(m,n)}=2n\pi i-(m\pi)^2$ we get

$$X_m(x)=\sin m\pi x,\quad Y_n(y)=e^{2n\pi i y}\quad (m,n\in \mathrm{Z}/\{0\}). \qquad (4.4)$$

Hence, non-trivial solution of homogeneous problem C in the domain $D_1$ has the form of

$$u_{mn}^{-}(x,y)=X_m(x)Y_n(y)=e^{2n\pi i y}\sin m\pi x$$

Similarly we can find non-trivial solutions of homogeneous problem C $u_{mn}^{+}(x,y)=X_m(x)Y_n(y)$ at $\lambda_2^{(m,n)}=(1+2n)\pi i-(m\pi)^2$ in the domain $D_2$, in which functions $X_m(x), Y_n(y)$ have the form of

$$X_m(x)=\sin m\pi x,\quad Y_n(y)=e^{(1+2n)\pi i y}\quad (m,n\in \mathrm{Z}/\{0\}).$$

One can see that $\lambda_i\,(i=1,2)$ located outside of the domain $\Delta$ ( see the Corollary 1).

**Corollary 2.** $\lambda_1^{(m,n)} = 2n\pi i - (m\pi)^2$, $\lambda_2^{(m,n)} = (1+2n)\pi i - (m\pi)^2$ *are eigenvalues, and* $u_{mn}^-(x,y) = e^{2n\pi iy}\sin m\pi x$, $u_{mn}^+(x,y) = e^{(1+2n)\pi iy}\sin m\pi x$, $(m,n \in Z/\{0\})$ *are eigenfunctions of the Problem C.*

We note that existence of the solution of Problem C can be proven similarly to the work [10].

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# Non-uniqueness of Transonic Flow past a Flattened Airfoil

**Alexander G. Kuz'min**[1]

[1]Institute of Mathematics and Mechanics
St Petersburg State University
28 University Ave., 198504 St Petersburg, Russia
alexander.kuzmin@pobox.spbu.ru

**ABSTRACT**

*A numerical analysis of inviscid and viscous transonic flows past a flattened airfoil is performed. Non-unique solutions of boundary value problems for the Euler and Navier-Stokes equations are obtained under certain conditions for flow parameters given on the far-field boundary.*

**Keywords:** transonic flow, airfoil, finite-volume solutions, stability, bifurcation.

**2000 Mathematics Subject Classification:** 76H05.

## 1 Introduction

Numerical studies of (Jameson, 1991), (Hafez and Guo, 1999a), (Hafez and Guo, 1999b) revealed non-unique steady solutions of boundary value problems for the system of Euler equations governing inviscid flow. A detailed analysis of the non-uniqueness was performed by (Ivanova and Kuz'min, 2004) and (Kuz'min, 2005) for flow past the airfoil

$$y(x) = \pm 0.06\sqrt{1-(2x-1)^4}\,(1-x^m)^2, \qquad 0 \le x \le 1. \tag{1.1}$$

It was shown that transonic solutions are unique if $m \le 8$. At the same time, if $8.5 \le m \le 12$ and, consequently, airfoil's curvature is small near the midchord $x = 1/2$, then there exist non-unique solutions for certain ranges of the angle of attack $\alpha$ and Mach number $M_\infty$ given on the far-field boundary.

In this paper, we analyze the non-uniqueness of transonic solutions for the airfoil ( with the parameter $m = 16$:

$$y(x) = \pm 0.06\sqrt{1-(2x-1)^4}\,(1-x^{16})^2, \qquad 0 \le x \le 1. \tag{1.1$'$}$$

In particular, we demonstrate that at $0.825 < M_\infty < 0.836,\ M_\infty \ne 0.833,\ \alpha = 0$ there exist both asymmetric and symmetric (with respect to the $x-$axis) steady solutions of the Euler equations. The realization of a particular solution depends on the time history of $M_\infty$ and $\alpha$.

## 2 Formulation of the problem and a numerical method

Let $D$ be a domain of the plane $(x, y)$ bounded by the airfoil (1.1′) and a $C-$type far-field boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_1$ is the vertical segment $x = 16,\ -15 \leq y \leq 15$, while the segments $\Gamma_2$ and $\Gamma_3$ are parallel to the freestream velocity vector, and $\Gamma_4$ is a circular arc (Fig. 1). The inviscid compressible adiabatic flow in the domain $D$ is governed by the system of Euler equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} &= 0, \\
\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial \rho u v}{\partial y} &= 0, \\
\frac{\partial \rho v}{\partial t} + \frac{\partial \rho u v}{\partial x} + \frac{\partial (\rho v^2 + p)}{\partial y} &= 0, \\
\frac{\partial e}{\partial t} + \frac{\partial (e + p) u}{\partial x} + \frac{\partial (e + p) v}{\partial y} &= 0,
\end{aligned} \tag{2.1}$$

where $p(x, y, t),\ \rho(x, y, t),\ u(x, y, t),\ v(x, y, t)$ are the static pressure, density, and components of the velocity in $x$ and $y$ directions, respectively; $e = p/(\gamma - 1) + \rho(u^2 + v^2)/2$ is the total energy per unit volume, $\gamma = 1.4$ is the ratio of specific heats.

A time-dependent solution $[p, \rho, u, v]$ of the system ( admitting shock waves and contact discontinuities is to be found in the domain $D$ under the initial-boundary conditions as follows. The angle of attack $\alpha = \mathrm{acrtan}(v_\infty/u_\infty)$ and the freestream Mach number $M = M_\infty < 1$ are given on the inflow part $\Gamma_4$ of $\Gamma$, while the static pressure $p = p_\infty$ is given on the outflow part $\Gamma_1$. The normal velocity component vanishes on the airfoil and on $\Gamma_2 \cup \Gamma_3$. The initial data are either the uniform flow determined by the freestream conditions or a steady flow obtained previously for other values of $M_\infty$ and $\alpha$. The Mach number is related to $[p, \rho, u, v]$ by the formula $M^2 = (u^2 + v^2)\rho/(\gamma p)$.

To obtain numerical solutions of the above initial-boundary value problem, we used the NSCKE finite-volume solver by (Mohammadi, 1994) in which the equations are discretized in space on an unstructured mesh using the Roe scheme. The second-order accuracy is recovered by means of a MUSCL reconstruction and Van Albada type limiters. The time integration was performed by an explicit four-stage Runge-Kutta scheme. Steady solutions $[\bar{p}, \bar{\rho}, \bar{u}, \bar{v}]$ of the boundary value problem were calculated using relaxation in time and the local time-stepping strategy,

$$p(x, y, t),\ \rho(x, y, t),\ u(x, y, t),\ v(x, y, t) \quad \rightarrow \quad \bar{p}(x, y),\ \bar{\rho}(x, y),\ \bar{u}(x, y),\ \bar{v}(x, y) \quad \text{as} \quad t \rightarrow \infty.$$

Computations were performed primarily on a nonuniform mesh of $733 \times 215$ grid points which were clustered near the airfoil and in the vicinity of shock waves. Each of the two sides of airfoil (1.1′) was represented by 265 nodes, whereas the outer boundary $\Gamma$ of the domain was discretized using 733 nodes. Test computations on a refined mesh yielded just marginal corrections in the obtained flow field. The high accuracy of the method was confirmed by application to transonic flow over the NACA 0012 airfoil at $\alpha = 1.25$ deg and comparison with results available in the literature (Delanyae, Geuzaine and Essers, 1997).

## 3 Results and discussion

Computation of a steady solution $[\bar{p}, \bar{\rho}, \bar{u}, \bar{v}]$ in the closed domain $\bar{D}$ permits one to find the lift coefficient via the expression $C_L = 2(\rho_\infty(u_\infty^2 + v_\infty^2))^{-1} \oint \bar{p}\cos(n, y)\,\mathrm{d}s$ in which the integration is carried out over the airfoil and $n$ is the normal vector. A numerical analysis has revealed that both $C_L$ and the structure of the obtained solutions depend crucially on the given parameters $\alpha$ and $M_\infty$.

In particular, the dependence of $C_L$ on $M_\infty$ at $\alpha = 0$ is displayed in Fig. 2. The left branch of the plot corresponds to the solutions $[\bar{p}, \bar{\rho}, \bar{u}, \bar{v}]_{2,2}$ which are symmetric with respect to the $x$-axis and exhibit two pairs of local supersonic regions. This branch was obtained by computing the steady solution with the aforementioned method at $M_\infty = 0.820,\ \alpha = 0,\ p_\infty = 26000$ N·m$^{-2}$ and then increasing the Mach number step-by-step to $0.832$. At each step, the previous steady flow was used as initial data. In particular, for $M_\infty = 0.827$ the obtained flow field is shown in Fig. 3,a.

The right branch of the plot displayed in Fig. 2 corresponds to the symmetric solutions $[\bar{p}, \bar{\rho}, \bar{u}, \bar{v}]_{1,1}$ with a single supersonic zone on each side of the airfoil. The solutions were obtained by computing the flow at $M_\infty = 0.842,\ \alpha = 0$ and then reducing the Mach number step-by-step to $0.8335$. In particular, for $M_\infty = 0.835$ the calculated flow field is shown in Fig. 4.

Between the right and left branches, there is a singular freestream Mach number $M_\mathrm{s} \approx 0.8325$ for which a symmetric solution cannot exist (see (Kuz'min and Ivanova, 2004) for an interpretation and details). As a consequence, at $M_\infty = M_\mathrm{s}$, time-dependent solutions $[p, \rho, u, v]$ converge to an asymmetric limit with three local supersonic regions, $[\bar{p}, \bar{\rho}, \bar{u}, \bar{v}]_{1,2}$ or $[\bar{p}, \bar{\rho}, \bar{u}, \bar{v}]_{2,1}$ (Fig. 5).

The obtained asymmetric solution persists when $M_\infty$ varies in the interval

$$0.825 < M_\infty < 0.836. \tag{3.1}$$

At the same time, if $M_\infty$ drops slightly below $0.825$ then one observes an abrupt transition to a symmetric solution with two pairs of local supersonic regions $[\bar{p}, \bar{\rho}, \bar{u}, \bar{v}]_{2,2}$. On the other hand, if $M_\infty$ increases beyond 0.836 then one observes a transition from the asymmetric solution to a steady symmetric solution with a single supersonic region on each side of the airfoil.

The symmetric and asymmetric solutions which correspond to the plot displayed in Fig. 2 are stable with respect to sufficiently small perturbations of the angle of attack. Meanwhile, if the perturbations exceed certain magnitudes, then one obtains a crucial restructuring of the flow field. For the steady symmetric flow, e.g., at $M_\infty = 0.827,\ \alpha = 0$ (Fig. 3,a), the perturbations $\alpha \geq 0.05$ deg trigger the coalescence of supersonic regions on the upper side of airfoil and the convergence of time-dependent solutions $[p, \rho, u, v]$ to an asymmetric solution $[\bar{p}, \bar{\rho}, \bar{u}, \bar{v}]_{1,2}$ as $t \to \infty$. The obtained flow pattern with a single supersonic region on the upper side persists if $\alpha$ is reset to zero (see Fig. 3,b). This corresponds to the transition from point $A_1$ to point $A_2$ residing on the upper branch of the plot depicted in Fig. 2.

Figure 6 demonstrates domains of the plane $(M_\infty, \alpha)$ in which the corresponding steady solutions are stable and their structures are preserved (Fig. 6). The domains overlap revealing a hysteresis with respect to variations of $\alpha$ and $M_\infty$. In the subdomains shown in light shade, there exist two steady solutions, while in the subdomains shown in dark shade, there exist three steady solutions of the boundary value problem formulated in Section 2. The realization of a particular solution depends on the time history of the given boundary conditions.

The maximum of $C_L$ and the length of the non-uniqueness interval ( are greater in the case under consideration as compared with that for airfoil ( with $m = 10$ and $m = 12$ studied by (Ivanova and Kuz'min, 2004) and (Kuz'min, 2005). This is accounted for by the smaller curvature of airfoil (1.1′) near $x = 1/2$ (see Fig. 7) and, consequently, a pronounced structural instability of the transonic flow.

Non-unique solutions were also revealed for the system of Reynolds-averaged Navier-Stokes equations which take into account the flow viscosity. In this case, the boundary condition on the airfoil is the no-slip condition $u = v = 0$. To close the system of governing equations, we used the Shear Stress Transport $k - \omega$ turbulence model (Menter, 1993). Numerical solutions exhibited two, three, or four local supersonic regions as well as flow oscillations caused by the boundary layer separation near airfoil's trailing edge. Magnitudes of the oscillations of $C_L(t)$ at $Re = 5.4 \times 10^6$ are indicated by the shaded subdomains in Fig. 8. The singular Mach number shifts from $0.833$ to $0.836$ in the case of viscous flow. Figure 9 demonstrates an instantaneous pattern of the Mach number isolines in the viscous flow with three local supersonic regions at $M_\infty = 0.832$.

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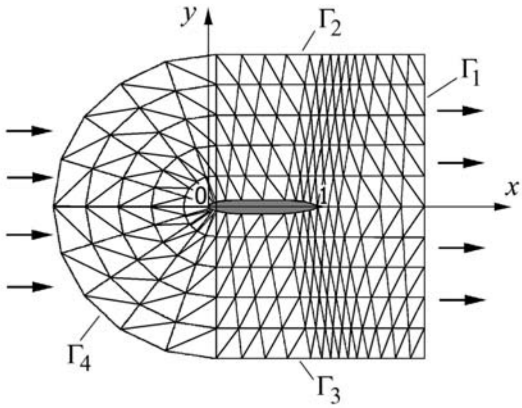


Figure 1. A sketch of the domain $D$ and the computational mesh.

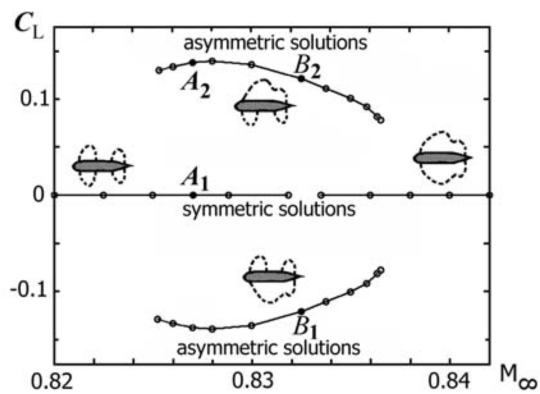


Figure 2. Lift coefficient $C_{\mathrm{L}}$ vs. $M_\infty$ at $\alpha = 0$ for steady inviscid flow past airfoil $(1.1')$.

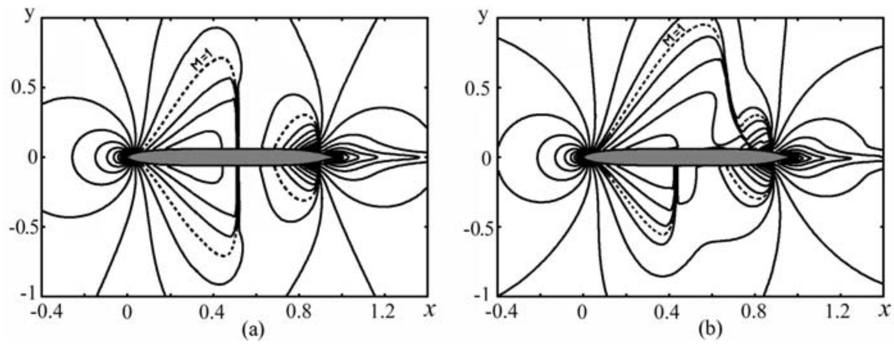


Figure 3. Mach number isolines for the steady inviscid flow at $M_\infty = 0.827,\ \alpha = 0$:
(a) symmetric flow corresponding to point $A_1$ in Fig. 2,
(b) asymmetric flow corresponding to point $A_2$ in Fig. 2.

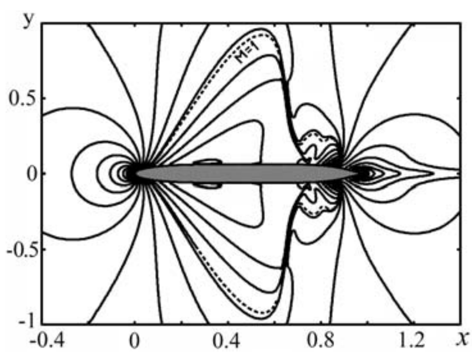


Figure 4. Mach number isolines for the steady symmetric flow at $M_\infty = 0.835,\ \alpha = 0$.

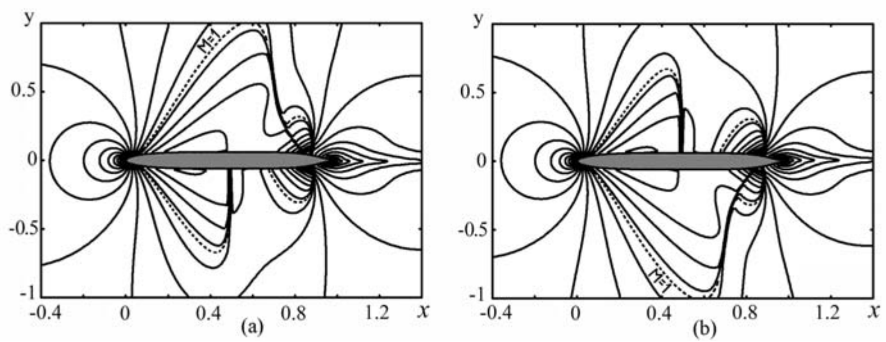


Figure 5. Mach number isolines for the steady flow at $M_\infty = 0.833,\ \alpha = 0$:
(a) asymmetric flow corresponding to point $B_2$ in Fig.2,
(b) asymmetric flow corresponding to point $B_1$ in Fig.2.

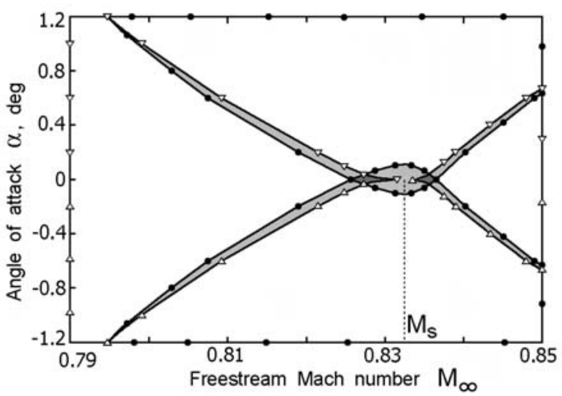


Figure 6. Subdomains of the non-uniqueness in the plane of freestream parameters.

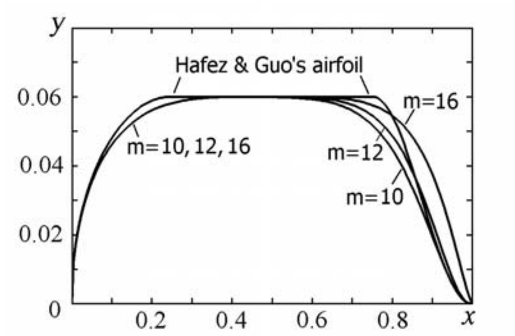


Figure 7. Upper parts of the airfoil (1.1) with $m = 10,\ 12,\ 16,$ and the flattened airfoil by Hafez and Guo (1999a).

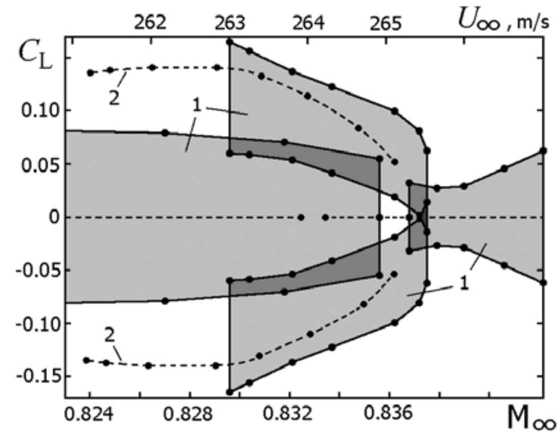


Figure 8. Lift coefficient $C_{\mathrm{L}}$ for airfoil (1.1′) at $\alpha = 0$:
1 - oscillating viscous flow, $Re = 5.4 \times 10^6$, 2 - steady inviscid flow (reproduced from Fig.2).

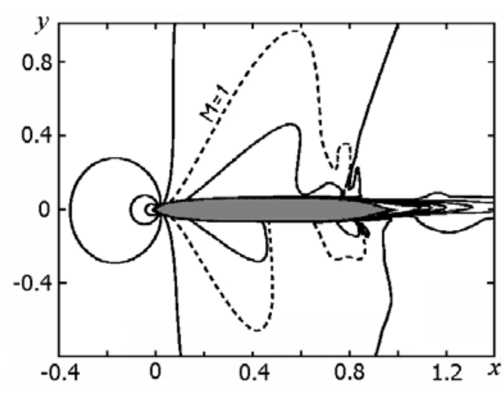


Figure 9. Instantaneous Mach number isolines for the asymmetric viscous flow at $M_\infty = 0.832,\ \alpha = 0,\ Re = 5.4 \times 10^6$.

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# Multiplier methods for mixed type equations

**Kevin R. Payne**

Dipartimento di Matematica "F. Enriques"
Universita' di Milano
Via Saldini, 50
20133 Milano, Italy
payne@mat.unimi.it

**ABSTRACT**

*We present a survey of recent results on existence, uniqueness and non-existence for boundary value problems for equations of mixed elliptic-hyperbolic type. The common technical feature is the use of suitable integral identities and estimates that arise from well chosen multipliers which are the infinitesimal generator of an invariance or almost invariance of the differential equation.*

**Keywords:** equations of mixed elliptic-hyperbolic type, multiplier methods, symmetry groups, existence, nonexistence.

**2000 Mathematics Subject Classification:** 35M10, 35A05, 35B33, 35B45.

## 1 Introduction

The use of multiplier methods as an organized tool in the study partial differential equations of mixed elliptic-hyperbolic type has its origins in the paper of Protter (1953), who, following the suggestion of Friedrichs, generalized the uniqueness proof of Frankl' (1945) for the Tricomi problem by reinterpreting the proof in the language of suitable energy integrals. Since its inception, this technique has become an indispensable tool due to its versatile technical power as well as due to its global nature, which allows for an a posteriori understanding as to which problems for mixed type equations are well posed. The problem of well-posedness is particularly difficult in this setting with its mixture of qualitative types and the use of type-dependent tools often results in a host of technical assumptions in order to glue together the pieces that come from dividing the problem along its type change interface. The purpose of this note is to give a survey of some recent developments in the use of multiplier methods for obtaining results on existence, uniqueness and nonexistence for linear and nonlinear problems of mixed type.

Before beginning our survey in earnest, we will briefly indicate what we mean by a *multiplier method* and in which class of *mixed type* equations we will work. By a multiplier method, we mean that given a partial differential equation

$$Lu = f \ \text{ in } \ \Omega \subset \mathbb{R}^N, \tag{1.1}$$

where $L$ is, for example, a second order linear differential operator with sufficiently smooth coefficients in $\Omega$ and $f$ a given function of $x$ (and perhaps $u$ or $\nabla u$ as well), one seeks to: 1) multiply (1.1) by a suitable *multiplier* $M[u]$; 2) integrate over $\Omega$; 3) manipulate the expression using integration by parts and eventual boundary conditions to arrive at a potentially useful integral identity. There is much artistry in both the choice of the multiplier $M[u]$ as well as in the choice of a suitable function space for $u$.

The class of mixed type equations we have in mind is that of *Chaplygin* type, in which the differential operator has the form

$$L = K(y)\Delta_x u + \partial_y^2 u, \tag{1.2}$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, $\Delta_x$ is the Laplace operator on $\mathbb{R}^n$ with $n \geq 1$ and the coefficient $K \in C^0(\mathbb{R})$ satisfies

$$K(0) = 0 \ \text{ and } \ K(y) \neq 0 \ \text{ for } \ y \neq 0 \tag{1.3}$$

so that the equation degenerates along the hypersurface $y = 0$. Our main interest concerns cases in which $K$ yields a change of type; that is, $K$ also satisfies

$$yK(y) > 0 \ \text{ for } \ y \neq 0, \tag{1.4}$$

so that the equation (1.1) is of mixed type (elliptic for $y > 0$ and hyperbolic for $y < 0$), although much of what will be discussed depends only on (1.2)-(1.3). The case $n = 1$ is particularly important due to its longstanding connection with transonic fluid flow (cf. (Frankl', 1945), (Bers, 1958), (Morawetz, 2004)) and also arises in problems of isometrically imbedding Riemannian manifolds whose Gauss curvature changes sign (cf. (Lin, 1986)).

As a final preparation for our survey, we give a brief history of the use of multiplier methods in mixed type equations. Protter's original work, cited above, made use of a *differential multiplier* $M[u] = a(x, y)u + b(x, y)u_x + c(x, y)u_y$ in treating the question of uniqueness of *quasi-regular* solutions to the *Tricomi problem* for an equation of Chaplygin type; that is,

$$Lu = K(y)u_{xx} + u_{yy} = f \ \text{ in } \ \Omega \tag{1.5}$$

$$u_{|\Gamma} = g, \tag{1.6}$$

where $\Gamma = \sigma \cup AC \subset \partial\Omega$ is particular proper subset of the boundary. More precisely, $\sigma$ is a piecewise regular curve in the elliptic region $\Omega^+$ (where $y > 0$), $AC$ is a characteristic of the operator $L$, and the *type change function* $K$ obeys (1.3)-(1.4) and certain technical conditions. Protter works directly on the second order equation and the function $u$ essentially lies in $C^2(\overline{\Omega})$. Using a suitable choice of the unknown coefficients $(a, b, c)$, integration by parts together with the vanishing of $u$ on $\sigma \cup AC$, yields an integral expression which is the sum of a positive definite quadratic form on $\Omega$ and boundary integrals which are non-negative, provided that the boundary satisfies a certain *starlike* condition, expressed by the non-negativity of a certain differential one form. In this way, technical boundary geometry restrictions enter into the hypotheses of the theorem. This result has been generalized to include other kinds of boundary conditions of the form $u = g$ on $\Gamma \subset \partial\Omega$ and to include additional classes of equations in a host of papers too numerous to mention.

Soon afterward, Morawetz (1958), by exploiting duality along the lines used by Friedrichs (1954) for hyperbolic equations, adapted the multiplier method in order to prove results on existence for problems such as (1.5)-(1.6). In her approach, the differential equation (1.5) is replaced by a more general first order system for $(v_1, v_2) = (u_x, u_y)$ of the form

$$\mathcal{L}v = \begin{bmatrix} K\partial_x & \partial_y \\ \partial_y & -\partial_x \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

The key step in her existence proof is a suitable a priori estimate for the formally skew-symmetric operator $\mathcal{L}$ which comes from: 1) defining a multiplier matrix $\mathcal{M}$ in the form

$$\mathcal{M} = \begin{bmatrix} b & c \\ -Kc & b \end{bmatrix},$$

with $\mathcal{M}$ invertible almost everywhere; 2) taking a test function $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^2)$ calibrated to the adjoint boundary conditions, and 3) estimating $(\mathcal{L}\varphi, \mathcal{M}^{-1}\varphi) = (\mathcal{L}\mathcal{M}\Phi, \Phi)$ from above and below, where

$$\Phi = \mathcal{M}^{-1}\varphi \;\text{ or }\; \varphi = \mathcal{M}\Phi.$$

A suitable multiplier pair $(b, c)$ can be chosen to give the non-negativity of the form $(\mathcal{L}\mathcal{M}\Phi, \Phi)$ if the boundary is suitably starlike with respect to $(b, c)$. Since $\mathcal{L}$ is formally skew-symmetric, one recognizes this argument as a uniqueness result for the adjoint problem. At the same time, and motivated in part by equations of mixed type, this technique was developed in a systematic way by Friedrichs (1958) in his discovery of positive symmetric systems. These two papers are the source of many important future developments, the former for attacks on problems related to transonic flow such as (Morawetz, 1956), (Morawetz, 1957) and the latter for its prominence in the theory of first order systems and as a touchstone in the development of type independent tools for partial differential equations (cf. (Lax and Phillips, 1960), (Phillips and Sarason, 1966), for example).

Shortly after the work of Morawetz, Berezanskii (1960), showed how one could use the multiplier technique directly on the second order equation to obtain a priori estimates on problems such as (1.5)-(1.6), and its adjoint problem, by estimating from above and below the quadratic forms $(L^*u, M[u])$ and $(Lu, M[u])$ with $u \in C^2(\overline{\Omega})$, calibrated to the boundary condition, and $M$ a suitable differential multiplier. The form of the estimate for $L^*$ is

$$||u||_{H^1_{\Gamma^*}} \leq C||L^*u||_{L^2}, \tag{1.7}$$

where $H^1_{\Gamma^*}$ is a Sobolev space of functions, calibrated to the adjoint boundary condition, which are square integrable together with their first derivatives, perhaps with suitable weights. Such an estimate on $L^*$ yields a uniqueness theorem for the adjoint problem as well as an existence theorem for the problem (1.5)-(1.6) and has been widely developed. With respect to the approach of Morawetz, the existence results obtained directly by this method are weaker, the solutions are guaranteed to lie only in $L^2$, while the forcing term $f$ can be more general (they lie the dual space $H^{-1}_{\Gamma^*}$). On the other hand, if one wants to treat the second order equation, this approach suggests the possibility of avoiding the passage to and return from the first order system.

On the other hand, Didenko (1973) introduced an important variant in the multiplier technique for problems such as (1.5)-(1.6), and its adjoint problem. His idea is to estimate from above and below the quadratic forms $(L^*u, M[u])$ and $(Lu, M[u])$, where $M[u]$ is now an integral multiplier. For example, in order to estimate the adjoint $L^*$, one starts from $u \in C_0^2(\Omega)$, and defines $v = M[u]$ as the solution of the first order Cauchy problem

$$\begin{cases} Mv := av + bv_x + cv_y = u \text{ in } \Omega \\ v = 0 \text{ on } \Gamma \end{cases}$$

where $\Gamma$ is portion of the boundary carrying the boundary condition (1.6). Such a technique, if it works, produces an a proiri estimate one step lower on the regularity scale with respect to (1.7), namely

$$||u||_{L^2} \leq C||L^*u||_{H_\Gamma^{-1}},$$

where $H_\Gamma^{-1}$ is the dual space to $H_\Gamma^1$ equipped with its *negative norm* in the sense of Lax. This approach retains the advantage of Berezanskii's method by working directly on the second order equation but it also produces solutions that are as strong as those obtained by Morawetz's technique.

There are two additional pieces of history which are important for what we have to say. First, in a ground breaking paper (1970), Morawetz was able to use her method to obtain the first result on well-posedness for weak solutions to the Dirichlet problem for mixed type equations. We call such a boundary value problem *closed* in the sense that the boundary condition is placed on the entire boundary, as opposed to an *open* boundary value problem in which the condition is placed on a proper subset of the boundary. As the uniqueness theorems suggest, a closed problem is typically over-determined in spaces of classical regularity. Such closed boundary value problems are important for transonic flows about profiles, while the open problems are linked to flows in nozzles (cf. (Bers, 1958), (Morawetz, 2004)). Morawetz's result (1970) is valid for the Tricomi equation ($K(y) = y$) and for domains which are lens-like and thin. Such restrictions are not particularly welcome in the applications to transonic flow since restrictions on boundary geometry correspond to restrictions on airfoil shape and the approximation $K(y) \sim y$ is valid only for nearly sonic speeds. Finally, we note that the method of multipliers has been used to prove uniqueness theorems for semilinear boundary value problems of mixed type, as done by Gvazava (1970), Rassias (1986) and others.

The recent progress we which to describe can be summarized in the following way. First, we identify a class of multipliers naturally associated to the differential operators of the form (1.2)-(1.4); namely, those which are the infinitesimal generators of the symmetry group for the differential operator. This group has been calculated in (Lupo and Payne, 2005) and will be discussed in section 2. Second, we apply variants of the methods of Berezanskii and Didenko to the Dirichlet problem in order to find robust results on the existence and uniqueness of weak solutions for such closed boundary value problems, which generalize the particular result of Morawetz (1970). These results have been proven in (Lupo, Morawetz and Payne, in press) and will be discussed in section 3. Finally, we give results on the nonexistence of nontrivial solutions for semilinear equations of mixed type with various boundary conditions. These are results from (Lupo and Payne, 2003) and (Lupo, Payne and Popivanov, 2006) and will be discussed in section 4. The key ingredient is our systematic use of multipliers which generate an invariance or almost invariance in the differential operator.

## 2 Symmetry groups and multiplier classes

In the art of choosing a useful multiplier for a given partial differential equation, it is a good idea to think of the complete symmetry group associated to the differential operator (or its principal part). The reason for this is fairly simple. If one has a continuous one parameter symmetry group of variational or divergence type associated to a differential equation

$$Lu = 0, \tag{2.1}$$

then Noether's theorem ensures the existence of a conservation law associated to the solutions of the differential equation; that is, an equation of the form

$$\operatorname{div} U = 0,$$

which must be satisfied by every sufficiently regular solution of (2.1), where $U$ is a nonlinear function of $(x, y, u, \nabla u)$. Integration of the conservation law over a domain then yields the vanishing of a certain boundary integral. The same integral identity results if one uses the multiplier method with $M[u]$ chosen as the *infinitesimal generator* of the one parameter symmetry group applied to $u$. Hence, in some sense, the simplest possible integral identities are those that result from an invariance property of the differential operator, or of its principal part.
We now recall briefly the results in (Lupo and Payne, 2005) on the symmetry groups for the equation (2.1) with $L$ of the form (1.2)-(1.4). The largest possible symmetry group results from the particular choice of power type degeneration

$$K(y) = y|y|^{m-1}, \;\; m > 0$$

(or $K(y) = \pm|y|^m$). The equation (2.1) is the Euler-Lagrange equation for the Lagrangian

$$\mathcal{L}(y, \nabla u) = \frac{1}{2}\left(K(y)|\nabla_x u|^2 + u_y^2\right). \tag{2.2}$$

One has, apart from certain trivial symmetries for these linear and homogeneous equations, symmetries coming from: 1) translations in the "space variables" $x$; 2) rotations in the space variables; 3) certain anisotropic dilations; 4) inversion with respect to a well chosen hypersurface (cf. Theorem 2.5 of (Lupo and Payne, 2005)).
More precisely, the trivial one parameter symmetry groups arise from the fact that if $u$ solves (2.1) then so does $u + \epsilon\beta$ with $\beta$ any solution of (2.1) and $\epsilon \in \mathbb{R}$. The non trivial symmetries are represented by the fact that if $u$ solves (2.1) with $K(y) = y|y|^{m-1}, m > 0$, then so do

$$u_{k;\epsilon}(x, y) = \mathcal{T}_{k;\epsilon}u(x, y) = u(x - \epsilon e_k, y) \tag{2.3}$$

$$u_{j,k;\epsilon}(x,y) = \mathcal{R}_{j,k;\epsilon}u(x,y) = u(A_{j,k;\epsilon}x,y) \tag{2.4}$$

$$u_\lambda(x,y) = \mathcal{S}_\lambda u(x,y) = \lambda^{-p(m,N)}u\left(\lambda^{-(m+2)}x,\lambda^{-2}y\right) \tag{2.5}$$

and

$$u_{k;\epsilon}(x,y) = \mathcal{I}_{k;\epsilon}u(x,y) = D_{k,\epsilon}^{-q(N,m)}u\left(\frac{x+\epsilon d e_k}{D_{k,\epsilon}},\frac{y}{D_{k,\epsilon}^{2/(m+2)}}\right), \tag{2.6}$$

where $\epsilon \in \mathbb{R}$ (and $|\epsilon|$ is small in $(2.6)$), $\lambda > 0$, $\{e_k\}_{k=1}^N$ is the standard basis of $\mathbb{R}^N$, $A_{j,k;\epsilon}$ is the $\epsilon$-rotation in the $x_j - x_k$ plane,

$$p(m,N) = \frac{N(m+2)-2}{2} > 0,$$

$$D_{k,\epsilon}(x,y) = 1 + 2\epsilon x_k + \epsilon^2 d(x,y),$$

where

$$d(x,y) = |x|^2 + \frac{4}{(m+2)^2}y|y|^{m+1}, \tag{2.7}$$

and

$$q(m,N) = \frac{N(m+2)-2}{2(m+2)} > 0.$$

The same result holds for the degenerate elliptic/hyperbolic cases where $K(y) = \pm|y|^m$ where it is enough to replace $y|y|^{m+1}$ with $\pm|y|^{m+2}$ in (2.7). The symmetries (2.3)-(2.5) are variational in the sense that they leave invariant the integral of the Lagrangian (2.2) while the symmetry (2.6) is a divergence symmetry and is only locally well defined. All yield associated conservation laws.

The infinitesimal generators of the nontrivial symmetries are given by the vector fields

$$v_k^T = \frac{\partial}{\partial x_k}, \quad k = 1,\ldots,N \tag{2.8}$$

$$v^D = (m+2)x\cdot\nabla_x + 2y\frac{\partial}{\partial y} - \frac{N(m+2)-2}{2}u\frac{\partial}{\partial u}, \tag{2.9}$$

$$v_{jk}^R = x_k\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_k}, \quad 1 \le j < k \le N \tag{2.10}$$

$$\begin{aligned} v_k^I \quad &= \quad -d(x,y)\frac{\partial}{\partial x_k} + 2x_k x\cdot\nabla_x + \frac{4}{m+2}x_k y\frac{\partial}{\partial y} \\ &\quad -\frac{N(m+2)-2}{m+2}x_k u\frac{\partial}{\partial u}, \quad k = 1,\ldots,N \end{aligned} \tag{2.11}$$

and together with the trivial symmetries generate the complete symmetry group (cf. Theorem 2.5 of (Lupo and Payne, 2005)). The proof of this claim exploits well known infinitesimal techniques for classifying the infinitesimal generators

$$v = \sum_{i=1}^N \xi^i(x,y,u)\frac{\partial}{\partial x_i} + \eta(x,y,u)\frac{\partial}{\partial y} + \varphi(x,y,u)\frac{\partial}{\partial u}$$

of symmetries, where $v$ is thought of as a vector field which acts an open subset $M$ of the *0-jet space*, $\mathbb{R}^{N+1} \times \mathcal{U}^{(0)} \simeq \mathbb{R}^{N+1} \times \mathbb{R}$ (the space of values for independent and dependent variables) together with the action of their *prolongations* onto higher order jet spaces (which includes the values of higher order derivatives of $u$). See the Appendix of (Lupo and Payne, 2005) for details. See also (Payne, in press) for the relation between the symmetry group and the class of conformal transformations with respect to a certain singular Riemannian metric associated to the operator $L$

Hence, we obtain a class of potentially useful differential multipliers of the form $M[u] = v(u)$ with $v$ one of the vector fields in (2.8)-(2.11) or a linear combination of them.

## 3 Well posedness for the Dirichlet problem

In this section, we examine the question of well-posedness for the Dirichlet problem for a second order linear partial differential equation of mixed elliptic-hyperbolic type. That is, given $f \in \mathcal{H}_0$, we ask if it is possible to show the existence of a unique $u \in \mathcal{H}_1$ which solves in some reasonable sense the problem

$$Lu = K(y)u_{xx} + u_{yy} = f \text{ in } \Omega \tag{3.1}$$

$$u = 0 \text{ on } \partial\Omega \tag{3.2}$$

where $\mathcal{H}_0, \mathcal{H}_1$ are functions spaces to be determined, $K \in C^1(\mathbb{R}^2)$ satisfies

$$K(0) = 0 \text{ and } yK(y) > 0 \text{ for } y \neq 0, \tag{3.3}$$

$\Omega$ is a bounded open and connected subset of $\mathbb{R}^2$ with piecewise $C^1$ boundary. We will assume throughout that

$$\Omega^{\pm} := \Omega \cap \mathbb{R}^2_{\pm} \neq \emptyset, \tag{3.4}$$

so that (3.1) is of mixed elliptic-hyperbolic type. We will call $\Omega$ a *mixed domain* if (3.4) holds.

As noted in the introduction, while the Dirichlet problem (3.1)-(3.2) is classically well-posed for an elliptic operator $L$, the presence of a hyperbolic subregion $\Omega^-$ leads to an over-determined problem in spaces of classical regularity. This phenomenon is well known for purely hyperbolic equations, as first noted by Picone (1910). From a suitable uniqueness theorem, one can show that if $u$ is a sufficiently smooth solution to (1.1) which vanishes only on a proper subset of the boundary $\Gamma = \partial\Omega \setminus \Sigma$, it must vanish on $\overline{\Omega}$. Hence, if one wants to impose the boundary condition on all of the boundary, one must expect in general that some real singularity must be present. Moreover, in order to prove well-posedness, one must make a good guess about where to look for the solution; that is, one must choose some reasonable function space which admits a singularity strong enough to allow for existence but not so strong as to lose uniqueness. This, in practice, has proven to be the main difficulty of the problem.

In order to present the well-posedness results for (3.1)-(3.2), we would like to first make precise the setting in which we will work. The function $K \in C^1(\mathbb{R})$ will be taken to satisfy (3.3) and

additional assumptions as necessary. In all that follows, $\Omega$ will be a bounded mixed domain (open, connected, satisfying (3.4)) in $\mathbb{R}^2$ with piecewise $C^1$ boundary so that we may apply the divergence theorem and $\nu$ will denote the external normal field. Since the differential operator (3.1) is invariant with respect to translations in $x$, we may assume that the origin is the point on the *parabolic line* $AB := \{(x,y) \in \overline{\Omega} : y = 0\}$ with maximal $x$ coordinate; that is, $B = (0,0)$. This will simplify certain formulas without reducing the generality of the results. We will sometimes require that $\Omega$ is *star-shaped with respect to the flow of a given (Lipschitz) continuous vector field* $V = (V_1(x,y), V_2(x,y))$; that is, for every $(x_0,y_0) \in \overline{\Omega}$ one has $\mathcal{F}_t(x_0,y_0) \in \overline{\Omega}$ for each $t \in [0,+\infty]$ where $\mathcal{F}_t(x_0,y_0)$ represents the time-$t$ flow of $(x_0,y_0)$ in the direction of $V$.
We will make use of several natural spaces of functions and distributions. We define $H_0^1(\Omega;K)$ as the closure of $C_0^\infty(\Omega)$ (smooth functions with compact support) with respect to the weighted Sobolev norm

$$||u||_{H^1(\Omega;K)} := \left[\int_\Omega \left(|K|u_x^2 + u_y^2 + u^2\right)\,dxdy\right]^{1/2}.$$

Since $u \in H_0^1(\Omega;K)$ vanishes weakly on the entire boundary, one has a *Poincarè inequality*: there exists $C_P = C_P(\Omega,K)$

$$||u||^2_{L^2(\Omega)} \leq C_P \int_\Omega \left(|K|u_x^2 + u_y^2\right)\,dxdy, \quad u \in H_0^1(\Omega;K). \tag{3.5}$$

The inequality (3.5) is proven in the standard way by integrating along segments parallel to the coordinate axes for $u \in C_0^1(\Omega)$ and then using continuity. An equivalent norm on $H_0^1(\Omega;K)$ is thus given by

$$||u||_{H_0^1(\Omega;K)} := \left[\int_\Omega \left(|K|u_x^2 + u_y^2\right)\,dxdy\right]^{1/2}. \tag{3.6}$$

We denote by $H^{-1}(\Omega;K)$ the dual space to $H_0^1(\Omega;K)$ equipped with its negative norm in the sense of Lax

$$||w||_{H^{-1}(\Omega;K)} := \sup_{0\neq\varphi\in C_0^\infty(\Omega)} \frac{|\langle w,\varphi\rangle|}{||\varphi||_{H_0^1(\Omega;K)}}, \tag{3.7}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket and one has the *generalized Schwartz inequality*

$$|\langle w,\varphi\rangle| \leq ||w||_{H^{-1}(\Omega;K)}||\varphi||_{H_0^1(\Omega;K)}, \quad w \in H^{-1}(\Omega;K),\ \varphi \in H_0^1(\Omega;K). \tag{3.8}$$

One clearly has a rigged triple of Hilbert spaces

$$H_0^1(\Omega;K) \subset L^2(\Omega) \subset H^{-1}(\Omega;K),$$

where the scalar product (on $L^2$ for example) will be denoted by $(\cdot,\cdot)_{L^2(\Omega)}$.
It is routine to check that the second order operator $L$ in (3.1) is formally self-adjoint when acting on distributions $\mathcal{D}'(\Omega)$ and gives rise to a unique continuous and self-adjoint extension

$$L : H_0^1(\Omega;K) \to H^{-1}(\Omega;K) \tag{3.9}$$

We will also make use of suitably weighted versions of $L^2(\Omega)$ and their properties. In particular, for $K \in C^1(\mathbb{R})$ satisfying (3.3) we define

$$L^2(\Omega; |K|^{-1}) := \{f \in L^2(\Omega) : \ \ |K|^{-1/2} f \in L^2(\Omega)\},$$

equipped with its natural norm

$$||f||_{L^2(\Omega;|K|^{-1})} = \left[\int_\Omega |K|^{-1} f^2 \, dxdy\right]^{1/2},$$

which is the dual space to the weighted space $L^2(\Omega; |K|)$ defined as the equivalence classes of square integrable functions with respect to the measure $|K|\, dxdy$; that is, with finite norm

$$||f||_{L^2(\Omega;|K|)} = \left[\int_\Omega |K| f^2 \, dxdy\right]^{1/2}.$$

One has the obvious chain of inclusions

$$L^2(\Omega; |K|^{-1}) \subset L^2(\Omega) \subset L^2(\Omega; |K|),$$

where the inclusion maps are continuous and injective (since $K$ vanishes only on the parabolic line, which has zero measure).

As a first step, using standard functional analytic techniques, one can obtain results on distributional existence and strong uniqueness for solutions to the Dirichlet problem (3.1)-(3.2). The key point is to obtain a suitable *a priori* estimate by adapting the multiplier method of Berezanskii to this closed boundary value problem and selecting the test function to have compact support. One obtains the following result

**Theorem 3.1.** *Let $\Omega$ be any bounded region in $\mathbb{R}^2$ with piecewise $C^1$ boundary. Let $K \in C^1(\mathbb{R})$ be a type change function satisfying (3.3) and*

$$K' > 0 \tag{3.10}$$

$$\exists \delta > 0 : \ \ 1 + \left(\frac{2K}{K'}\right)' \geq \delta \tag{3.11}$$

**a)** *There exists a constant $C_1(\Omega, K)$ such that*

$$||u||_{H^1_0(\Omega;K)} \leq C_1 ||Lu||_{L^2(\Omega)}, \ \ u \in C^2_0(\Omega). \tag{3.12}$$

**b)** *For each $f \in H^{-1}(\Omega; K)$ there exists $u \in L^2(\Omega)$ which distributionally solves (3.1)-(3.2) in the sense that*

$$(u, L\varphi)_{L^2} = \langle f, \varphi\rangle, \ \ \varphi \in H^1_0(\Omega; K) : \ L\varphi \in L^2(\Omega). \tag{3.13}$$

This theorem is the union of Lemma 2.1 and Theorem 2.2 of (Lupo et al., in press), where the complete proof is given. It should be noted that, since $L$ is formally self-adjoint, the estimate (3.12) also holds for $L^t = L$ and that, in (3.13), $L$ is the self-adjoint extension (3.9). The estimate (3.12) follows from the multiplier method in which one considers an arbitrary $u \in$

$C_0^2(\Omega)$ and seeks to estimate the expression $(Mu, Lu)_{L^2}$ from above and below where $Mu = au + bu_x + cu_y$ is the multiplier to be determined. Using

$$a \equiv -1, \quad b \equiv 0, \quad c = c(y) = \max\{0, -4K/K'\} \tag{3.14}$$

one has the needed positive lower bound, while the Cauchy-Schwartz inequality is used for the upper bound. The two estimates are combined with the Poincarè inequality (3.5) to complete the estimate (3.12). The proof of the existence in part **b** is a standard argument using the Hahn-Banach theorem and the Riesz representation theorem.

The estimate (3.12) also shows that sufficiently strong solutions must be unique. We say that $u \in H_0^1(\Omega; K)$ is a *strong solution* of the Dirichlet problem (3.1)-(3.2) if there exists an approximating sequence $u_n \in C_0^2(\Omega)$ such that

$$||u_n - u||_{H^1(\Omega;K)} \to 0 \text{ and } ||Lu_n - f||_{L^2(\Omega)} \to 0 \text{ as } n \to +\infty$$

The following theorem is an immediate consequence of the definition.

**Theorem 3.2.** *Let $\Omega$ be any bounded region in $\mathbb{R}^2$ with piecewise $C^1$ boundary. Let $K \in C^1(\mathbb{R})$ be a type change function satisfying (3.3), (3.10) and (3.11). Then any strong solution of the Dirichlet problem (3.1)-(3.2) must be unique.*

*Remarks:*

1. The class of admissible $K$ is very large and includes the standard models for transonic flow problems such as the Tricomi equation with $K(y) = y$ and the Tomatika-Tamada equation $K(y) = A(1 - e^{-2By})$ with $A, B > 0$ constants.

2. The result also holds for non strictly monotone functions such as the Gellerstedt equation with $K(y) = y|y|^{m-1}$ where $m > 0$. In this case, one can check that in place of (3.14) it is enough to choose the *dilation multiplier*, introduced in (Lupo and Payne, 2003), whose principal part was given in (2.9)

$$a \equiv 0, \quad b = (m+2)x, \quad c = 2y.$$

3. No boundary geometry hypotheses have yet been made; in particular, there are no star-like hypotheses on the elliptic part and no sub-characteristic hypotheses on the hyperbolic part. These kinds of hypotheses will enter when we look for solutions in a stronger sense.

It is clear that the distributional existence is in a very weak sense; too weak, in fact, to be very useful. In particular, the sense in which the solution vanishes at the boundary is only by duality and one may not have uniqueness. Example 2.4 of (Lupo et al., in press) gives one way in which things may go wrong. In any case, the existence result is a first general indication that while the closed Dirichlet problem is generically over-determined for regular solutions, it is generically not over-determined if one looks for a solution which is taken in a sufficiently weak sense. Moreover, while uniqueness generically holds for strong solutions, one must show that such strong solutions exist.

We are now ready for the well-posedness result, which shows that there is a way to steer a course between the weak existence and the strong uniqueness result for the Dirichlet problem by following the path laid out by Didenko (1973) for open boundary value problems. The suitable notion of solutions (called generalized solutions by Didenko in the case of open boundary conditions) is contained in the following definition.

**Definition 3.3.** We say that $u \in H_0^1(\Omega; K)$ *is a weak solution of the Dirichlet problem (3.1)-(3.2)* if there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$||u_n - u||_{H_0^1(\Omega;K)} \to 0 \text{ and } ||Lu_n - f||_{H^{-1}(\Omega;K)} \to 0, \text{ for } n \to +\infty$$

or equivalently

$$\langle Lu, \varphi \rangle = -\int_\Omega (K u_x \varphi_x + u_y \varphi_y)\, dxdy = \langle f, \varphi \rangle, \ \ \varphi \in H_0^1(\Omega, K),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $H_0^1(\Omega; K)$ and $H^{-1}(\Omega; K)$, $L$ is the continuous extension defined in (3.9), and the relevant norms are defined in (3.6) and (3.7).
Our first result concerns the Gellerstedt operator; that is, with $K$ of pure power type

$$K(y) = y|y|^{m-1}, \ \ m > 0. \tag{3.15}$$

**Theorem 3.4.** *Let $\Omega$ be a bounded mixed domain with piecewise $C^1$ boundary and parabolic segment $AB$ with $B = 0$. Let $K$ be of pure power form (3.15). Assume that $\Omega$ is star-shaped with respect to the vector field $V = (-(m+2)x, -\mu y)$ where $\mu = 2$ for $y > 0$ and $\mu = 1$ for $y < 0$. Then*
**a)** *there exists $C_1 > 0$ such that*

$$||u||_{L^2(\Omega;|K|)} \leq C_1 ||Lu||_{H^{-1}(\Omega;K)}, \ \ u \in C_0^\infty(\Omega). \tag{3.16}$$

**b)** *for each $f \in L^2(\Omega; |K|^{-1})$ there exists a unique weak solution $u \in H_0^1(\Omega; K)$ in the sense of Definition 3.3 to the Dirichlet problem (3.1)-(3.2).*

Note that the theorem allows for both non lens-like and lens-like domains and hence relaxes substantially the restrictions on the geometry of the boundary with respect to the original result of Morawetz (1970). Moreover, the star-shaped assumption implies that the hyperbolic boundary is non-characteristic.
This theorem is the union of Lemma 3.3 and Theorem 3.2 of (Lupo et al., in press). The existence in part **b)** follows from the a priori estimate of in much the same way as in Theorem 2.1. The uniqueness follows also from the estimate and the Definition 2.3. To obtain the estimate (3.16), the basic idea is to adapt the Didenko multiplier method to this closed boundary value problem; that is, to estimate from above and below the expression $(Iu, Lu)_{L^2(\Omega)}$ for each $u \in C_0^\infty(\Omega)$ where $v = Iu$ is the solution to the following auxiliary Cauchy problem

$$\begin{cases} Mv := av + bv_x + cv_y = u \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega \setminus B \end{cases}$$

where $B = (0, 0)$ is the righthand endpoint of the parabolic line and

$$a \equiv -1/4, \ \ (b, c) = -V = ((m+2)x, \mu y). \tag{3.17}$$

This multiplier is almost the principal part the dilation and translation multipliers (2.9); it has been fudged a bit in the hyperbolic region in order to obtain the necessary positivity in a key step of the proof. One analyzes first the properties of the solution $v$, which is shown to lie in $C^{\infty}(\Omega^{\pm}) \cap C^0(\overline{\Omega}) \cap H^1_0(\Omega; K)$. Then one estimates from below where the choice (3.17) ensures the positivity of the quadratic form $(Iu, Lu)$. The rest of the estimate proceeds as before, using the generalized Schwartz inequality (3.8), continuity properties of the differential operator $M$, and the Poincarè inequality (3.5).

*Remarks:*

1. One can eliminate almost entirely the boundary geometry restrictions in the elliptic part of the domain by patching together the a priori estimate (3.16) with an easy estimate on elliptic subdomains using the multiplier $Mu = u$ (see Theorem 3.4 of (Lupo et al., in press)).
2. One can replace the type change functions $K$ of pure power type with more general forms in which there is a bound on the variation of $(K/K')'$ for $y$ small (see Proposition 3.5 of (Lupo et al., in press)).
3. Moreover, it should be possible to relax the regularity assumptions on the type change function $K$. For example, this should be the case for the Lavrentiev-Bitsadze equation in which $K(y) = \mathrm{sign}(y)$ (Lavrentiev and Bitsadze, 1950). This equation has been well studied for problems with open boundary conditions; for example, see (Rassias, 1990) for techniques along the lines of Protter and Berezanskii.
4. With respect to the original result of Morawetz, our improvements are due in part from the fact that we work directly on the second order equation instead of reducing to a first order system. Working with the equation allows for a greater freedom in choosing the multipliers $(a, b, c)$; for the first order system there is no coefficient corresponding to $a$.
5. The norm $H^1(\Omega; K)$ employed here has a weight $|K|$ which vanishes on the entire parabolic line and hence one might worry that the solution is not locally $H^1$ due to the term $|K|u_x^2$. In fact, the solution does lie in $H^1_{\mathrm{loc}}(\Omega)$ as is shown in section 5 of (Lupo et al., in press), which contains the beginnings of a regularity theory. On the other hand, the norms used in (Morawetz, 1970) for treating the equation by way of a first order system were carefully constructed so as not to have weights vanishing on the interior.
6. In (Lupo et al., in press), we have also established well-posedness for closed problems with mixed boundary conditions; Dirichlet conditions on the elliptic boundary and *conormal* conditions on the hyperbolic boundary. The conormal condition is natural for transonic flow problems when the unknown is a perturbation of the velocity potential. On the other hand, for the Lavrentiev-Bitsadze operator, Pilant (1985), has obtained a result similar in spirit to that of Morawetz (1970) for the full conormal problem. His result follows the idea of Morawetz and has similarly strong restrictions on the geometry of the boundary, which must be lens-like and thin. Attempts to handle general classes of equations in general domains for the full conormal problem are in progress.

## 4 Nonexistence for semilinear problems

It is well known, starting from the seminal paper of Pohožaev (1965), that the homogeneous Dirichlet problem for semilinear elliptic equations such as $\Delta u + u|u|^{p-2} = 0$ in $\Omega$ a bounded subset of $\mathbb{R}^n$, with $n \geq 3$, will permit only the trivial solution $u \equiv 0$ if the domain is star-shaped, the solution is sufficiently regular, and $p > 2^*(n) = (n+2)/(n-2)$ the critical exponent in the Sobolev embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for $p \leq 2^*(n)$ which fails to be compact at the critical exponent. The proof of this result hinges upon a multiplier method in which one uses the infinitesimal generator of the invariance of the Laplace operator with respect to dilations. Hence, there arises the natural question of whether an analogous result holds for equations of mixed type. That is, if $L$ is a mixed type operator of Gellerstedt type

$$L = K(y)\Delta_x u + \partial_y^2 u, \quad K(y) = y|y|^{m-1} \text{ with } m > 0, \tag{4.1}$$

one would like to know if sufficiently smooth solutions to the problem

$$Lu + F'(u) = 0 \text{ in } \Omega \tag{4.2}$$

$$u = 0 \text{ on } \Sigma \subseteq \partial\Omega \tag{4.3}$$

must vanish identically in $\overline{\Omega}$ if $F$ a nonlinearity with a suitable notion of supercritical growth. The answer is yes, as has been shown in (Lupo and Payne, 2003) and (Lupo et al., 2006). The proof for the closed Dirichlet problem ($\Sigma = \partial\Omega$) is almost completely analogous with the classical elliptic case, while for open boundary value problems, there is a critical difference in that one must be able to control the sign of additional boundary integrals. In many cases, we are able to control these terms by using a suitable Hardy-Sobolev inequality.
In order to state the main results, we need to recall a few notions. Here $\Omega \subset \mathbb{R}^{N+1}$ will again be bounded with a piecewise $C^1$ boundary. The hyperbolic boundary $\Sigma_- = \partial\Omega \cap \mathbb{R}^{N+1}_-$ will be called *sub-characteristic* for $L$ if one has

$$K(y)|\nu_x|^2 + \nu_y^2 \geq 0, \text{ on } \Sigma_- \tag{4.4}$$

where $\nu = (\nu_x, \nu_y)$ is the (external) normal field on the boundary. This normal field is well defined with the possible exception of a finite number of points which will create no essential difficulty. If the inequality (4.4) holds in the strict sense, we will call $\Sigma_-$ *strictly sub-characteristic* which just means that $\Sigma_-$ is a piece of a *spacelike hypersurface* for the operator $L$ which is hyperbolic for $y < 0$. The operator $L$ in (4.1) is invariant with respect to the anisotropic dilation whose infinitesimal generator is

$$V = -\sum_{j=1}^{N}(m+2)x_j\partial_{x_j} - 2y\partial_y \tag{4.5}$$

(cf. (2.14)). This Lipschitz continuous vector field $V$ defines the class of domains which are admissible for the nonexistence principle; namely, those that are $V$-star-shaped, as defined in section 3. Furthermore, we recall that if $\Omega$ is $V$-star-shaped then $\partial\Omega$ will be *$V$-star-like* in the sense that on $\partial\Omega$ one has

$$(\alpha, \beta) \cdot \nu \geq 0 \tag{4.6}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\nu$ is the external unit normal to $\partial\Omega$ (cf. Lemma 2.2 of (Lupo and Payne, 2003)). If the inequality (4.6) holds in the strict sense, we will say that $\partial\Omega$ is *strictly $V$-star-like*. The differential operator $L$ is also invariant with respect to translations in the $x$ variables, so one may normalize a given problem by assuming a particular location of the origin without loss of generality, as done in section 3.

The dilation generated by $V$ in (4.5) also gives rise to a critical exponent

$$2^*(N, m) = \frac{2[N(m+2)+2]}{N(m+2)-2} \tag{4.7}$$

for the embedding of the weighted Sobolev space $H_0^1(\Omega; m)$ into $L^p(\Omega)$ where

$$||u||^2_{H_0^1(\Omega;m)} := \int_\Omega \left(|y|^m |\nabla_x u|^2 + u_y^2\right) \, dxdy$$

defines a natural norm for which to begin the search for weak solutions (cf. Proposition 2.4 of (Lupo and Payne, 2003) and section 3 of (Lupo and Payne, 2005)). More precisely, this norm is the natural norm for variational solutions where one notes that the equation (4.2) is the Euler-Lagrange equation associated to the functional $J(u) = \int_\Omega (\mathcal{L}(y, \nabla u) + F(u)) \, dxdy$ where $\mathcal{L}(y, \nabla u)$ is the Lagrangian defined in (2.4). One has the following result (cf. Theorem 2.1 of (Lupo et al., 2006)).

**Theorem 4.1.** *Let $\Omega$ be mixed type domain which is star-shaped with respect to the generator $V$ of the dilation invariance defined in (4.5) and whose hyperbolic boundary is sub-characteristic in the sense (4.4). Let $u \in C^2(\overline{\Omega})$ be a solution to (4.1)-(4.3) with $F'(u) = u|u|^{p-2}$. If $p > 2^*(N, m)$ the critical Sobolev exponent (4.7), then $u \equiv 0$. If, in addition, $\partial\Omega$ is strictly $V$-star-like and $\Sigma_-$ is strictly sub-characteristic, then the result holds also for $p = 2^*(N, m)$.*

We recall only the outline of the proof. Consider the primitive $F$ satisfying $F(0) = 0$. One multiplies (4.2) by $Mu = -Vu$ and integrates by parts and uses the boundary condition $u = F(u) = 0$ on $\partial\Omega$ to find the Pohožaev type identity calibrated to the dilation invariance

$$\int_\Omega \left[(N(m+2)+2)F(u) - \frac{(N(m+2)-2)}{2} uF'(u)\right] \, dxdy = \int_{\partial\Omega} W \cdot \nu \, d\sigma \tag{4.8}$$

where

$$W = [(m+2)x \cdot \nabla_x u + 2yu_y](K\nabla_x u, u_y) - \mathcal{L}((m+2)x, 2y)$$

and $\mathcal{L}$ is the Lagrangian defined in (2.2). In the supercritical case $p > 2^*(N, m)$, if $u$ is nontrivial, then one arrives at a contradiction as the left hand side of (4.8) is negative, while the right hand side is non-negative if $u$ vanishes on the boundary which is assumed starlike and sub-characteristic. In the critical case $p = 2^*(N, m)$, if the boundary is strictly starlike, then the identity (4.8) shows that the normal derivative of $u$ vanishes on the boundary (together with $u$). Another multiplier identity using $M[u] = u_y$ then shows that $u$ vanishes on $\overline{\Omega}$.

We now turn our attention to the more difficult case in which the boundary condition is placed on a proper subset $\Sigma$ of the boundary $\partial\Omega$. As a rule of thumb, $\Sigma$ will be chosen in order to yield a uniqueness theorem for the linear version of the problem. After all, the nonexistence of nontrivial solutions for the semilinear problem can be viewed as a uniqueness theorem. For simplicity, we will describe the simplest case, namely the Tricomi problem in dimension two, but we will make some remarks about generalizations at the end of the section. We consider the problem

$$K(y)u_{xx} + u_{yy} + F'(u) = 0 \text{ in } \Omega \tag{4.9}$$

$$u = 0 \text{ on } \Sigma = \sigma \cup AC, \tag{4.10}$$

where $K(y) = y|y|^{m-1}$ with $m > 0$, $\sigma$ is an arc in the elliptic region and $AC, BC$ are characteristics of negative, positive slope for $L = K(y)\partial_x^2 + \partial_y^2$. Again, without loss of generality we may assume that the point $B = (0,0)$ and hence the equation for the characteristic $BC$ becomes

$$BC = \{(x,y) \in \mathbb{R}^2 \mid y \le 0;\ (m+2)x = -2(-y)^{(m+2)/2}\}. \tag{4.11}$$

We have proven the following result (cf. Theorem 4.2 of (Lupo and Payne, 2003)).

**Theorem 4.2** *Let $\Omega \subset \mathbb{R}^2$ be a Tricomi domain with boundary $\sigma \cup AC \cup BC$. Assume that $\Omega$ is star-shaped with respect to the generator $V$ of the dilation invariance for $L$. Let $u \in C^2(\overline{\Omega})$ be a solution to (4.9)-(4.10) with $F'(u) = u|u|^{p-2}$. If $p > 2^*(1,m) = 2(m+4)/8$ the critical Sobolev exponent, then $u \equiv 0$.*

We recall the outline of the proof. The Pohožaev identity calibrated to the dilation invariance in this case is (cf. Theorem 3.1 of (Lupo and Payne, 2003)):

$$\int_\Omega \left[(m+2)F(u) - \frac{m}{2}uF'(u)\right]\, dxdy = \int_\sigma W_1 \cdot \nu\, ds + \int_{BC} (W_1 + W_2) \cdot \nu\, ds \tag{4.12}$$

where

$$W_1 = [(m+2)xu_x + 2yu_y](Ku_x, u_y) - \frac{1}{2}[Ku_x^2 + u_y^2]((m+2)x, 2y)$$
$$W_2 = F(u)((m+2)x, 2y) + \frac{m}{2}u(Ku_x, u_y).$$

For supercritical $p > 2^*(1,m)$, the integral over $\Omega$ is negative for nontrivial $u$. On the other hand, the first boundary integral in (4.12) is non-negative due to the boundary condition and the starlike hypothesis on $\sigma$. The second boundary integral is “new” and reflects the fact that the boundary condition has not been placed on $BC$. After choosing the parametrization $\gamma(t)$ of the characteristic $BC$ defined in (4.11) with $t$ equals $y$ as parameter, integration by parts shows that

$$\int_{BC} (W_1 + W_2) \cdot \nu\, ds = \int_{y_C}^0 \left[4(-t)^{(m+2)/2}\psi'(t)^2 - \frac{m^2}{4}(-t)^{(m-2)/2}\psi^2(t)\right]\, dt \tag{4.13}$$

where $\psi(t) := u(\gamma(t)) \in C^2((y_C, 0)) \cap C^1([y_C, 0])$. The non-negativity of (4.13) is equivalent to the validity of a Hardy-Sobolev inequality for $\psi$ of the form

$$\int_{y_C}^{0} \psi^2(t) w(t)\, dt \leq C_L^2 \int_{y_C}^{0} (\psi'(t))^2 v(t)\, dt$$

with weights $w(t) = (-t)^{(m-2)/2}$ and $v(t) = (-t)^{(m+2)/2}$ where one also needs that the best constant $C_L$ in the inequality satisfies $C_L^2 \leq 16/m^2$. Lemma 4.3 of (Lupo and Payne, 2003) provides exactly this result which is a transcription of a result of Opic and Kufner (Theorem 1.14 of (Opic and Kufner, 1990)). This completes the proof for supercritical $p$.

Remarks:

1. Theorem 4.2 also holds for other two dimensional problems with suitable choices for $\Omega$ and $\Sigma$. For example, by replacing the characteristic $AC$ with a sub-characteristic arc $\Gamma_1$, one arrives at the so-called *Frankl' problem*, where $\Gamma = \sigma \cup \Gamma_1$. For supercritical growth $p > 2^*(1,m)$, one has Theorem 4.2 (cf. Theorem 3.1 of (Lupo et al., 2006)). The same consideration holds for the so-called *Guderley-Morawetz* problem in which $\Gamma$ is taken as the part of $\partial\Omega$ outside of a backward light cone with vertex at the origin, assuming that the reminder of the boundary is $V$-starlike and its hyperbolic part is sub-characteristic.

2. If $\Gamma$ is strictly $V$-starlike with its hyperbolic part strictly sub-characteristic, then Theorem 4.2 also holds at critical growth $p = 2^*(1,m)$, using ideas similar to those used in the Dirichlet problem at critical growth (cf. Theorem 3.1 of (Lupo et al., 2006)).

3. In higher dimensions, one can prove analogous results under suitable conditions. For example, one can treat the so-called *Protter problem* in $\Omega \subset \mathbb{R}^{N+1}$ for the operator (4.1) (cf. Theorem 4.1 of (Lupo et al., 2006)). Here the hyperbolic boundary is formed by certain characteristic surfaces and was proposed by Protter (1954) as an analog of the Guderley-Morawetz and Tricomi problems in higher dimensions, although the analogy has proven to be weaker than what one would have expected (cf. (Popivanov, 1979) and (Rassias, preprint), for example).

4. One can also consider mixed type operators with substantially different qualitative structures. For example, one can prove nonexistence results for an operator with *lateral degeneration*, such as $L = K(y)\partial_{x_1}^2 = \partial_{x_2}^2$ in $\mathbb{R}^3$. See section 5 of (Lupo et al., 2006) for details.

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# Tricomi-Protter Problem of nD Mixed Type Equations

**John Michael Rassias**

Pedagogical Department
Section of Mathematics and Informatics
National and Capodistrian University of Athens
Attikis 15342 Athens , GREECE
e-mail: jrassias@primedu.uoa.gr; jrassias@tellas.gr

**Abstract**

*The Tricomi equation was established, in 1923, by F. G. Tricomi (Atti Accad. Naz. Lincei, 14, 133-247) who was the pioneer of parabolic elliptic and hyperbolic boundary value problems. In 1953, 1954 and 1955 M. H. Protter (J. Rat. Mech. Anal. : 2, 107-114, 1953 ; 3,435-446, 1954; 4, 721-732, 1955) generalized these problems even further for the Tricomi equation in three dimensions. In 1977 the author (Ph.D. Dissertation, Univ. Cal., Berkeley) generalized these results in n dimensions. In 2004, A. Kuzmin ,Frontiers of Fluid Mechanics (World Sci. Publ., Singapore, 285-295) considered the bifurcation of transonic flow over a flattened airfoil. In 2005, G. Wen (Applicable Analysis, 84(12), 1267-1286) investigated the Tricomi problem for second order linear equations of mixed type with parabolic degeneracy. In this paper we investigate the Tricomi- Protter problem of mixed type equations in n dimensions.*

**Key words and phrases:** Tricomi- Protter problem; Mixed type equation; Tricomi equation.

**AMS (MOS) Subject Classification:** 35M05

## 1. Introduction

In 1904 Chaplygin [10] noticed that the equation of a perfect gas was $K(y)u_{xx} + u_{yy} = 0$. In 1923 Tricomi [17] initiated the work on boundary value problems for linear partial differential mixed type equations of second order and related equations of variable type. In 1945 Frankl [1] drew attention to the fact that the Tricomi problem was closely connected to the study of gas flow with nearly sonic speeds.in 1953, and 1955 Protter [4,6] generalized and improved the afore mentioned results in the Euclidean plane. Furthermore Protter [5] he was

the first who investigated the Tricomi problem of 3D mixed type equations. In 1977 we [7] generalized Protter' s results in $\mathbb{R}^n (n > 2)$ for more general mixed type equations. In 1982 we [8] established a maximum principle of the Cauchy problem for hyperbolic equations in $\mathbb{R}^n (n \geq 2)$.In 1983 we [9] solved the Tricomi problem with two parabolic lines of degeneracy. In 1986 Kracht and Kreyszig [2]

established new interesting results in mixed type equations. Other interesting results in this area have been achieved by Wen and Begehr [14] in 1990, Semerdjieva [13] in 1993, Wen [15-16] in 2002 and 2005, Kuzmin [3] in 2005, as well as the author [10-12] in 1990, 1999 and 2002. In this paper we introduce the new nD (n>2) parabolic elliptic-hyperbolic partial differential equation

$$Lu \equiv K_1(t)(\sum_{i=1}^{n} u_{x_i x_i}) + (K_2(t)u_t)_t + r(x,t)u = f(x,t) \tag{*}$$

which is parabolic on $t=0$; elliptic in $t>0$ ;and hyperbolic in $t<0$, for $x=(x_1,x_2,...,x_n)$, and $K_2(0)>0$; $K_2(t)>0$ for all real $t\neq 0$, as well as $K_1(0)=0$; $K_1(t)>0$ for $t>0$;and $K_1(t)<0$ for $t<0$,as well as $(\ \ )_t = \partial/\partial t$. Besides we investigate the pertinent Tricomi - Protter problem of this equation. Furthermore we establish uniqueness of quasi-regular solutions for the afore-mentioned Tricomi – Protter problem. However the existence of weak solutions and well-posedness of regular solutions for this mixed type boundary value problem in several variables are still *open.* These research results are interesting in Fluid Mechanics.

## 2. Tricomi – Protter problem

Consider the parabolic elliptic-hyperbolic equation (*) in a bounded simply-connected mixed domain G with a piecewise smooth boundary $\partial G = S_1 \cup S_3 \cup S_4$, where $f=f(x,t)$ is continuous in G, $r=r(x,t)$ is once-continuously differentiable in G, $K_i = K_i(t)\ (i=1,2)$ are monotone increasing continuously differentiable for $t\in[-k_1,k_2]$ with $-k_1=\inf\{t:(x,t)\in G\}$ and $k_2=\sup\{t:(x,t)\in G\}$, for $x=(x_1,x_2,...,x_n)$.Furthermore we denote

$$K_1(t) = K(t)K_2(t)$$

for all *(x, t)* in G.

The boundary $\partial G$ of a bounded simply-connected region G of $\mathbb{R}^{n+1} (n>2)$ is formed by the following surfaces:

(1) A piecewise smooth surface $S_1 : \chi(x,t)=0$ lying in the elliptic region $G^+ = \{(x,t)\in G : t>0\}$ which intersects the plane $G_0 = \{(x,t)\in G : t=0\}$ in $|x|=1$; and (2) two characteristic surfaces $S_3$ and $S_4$ of (*):

$$S_3 : \Psi(x,t) \equiv |x| - 1 + \rho = 0, \text{ and } S_4 : \Phi(x,t) \equiv |x| - \rho = 0,$$

lying in the hyperbolic region $G^- = \{(x,t)\in G : t<0\}$ and satisfying the conditions

$$K_1(t)[\sum_{i=1}^{n}(\partial\Psi/\partial x_i)^2] + K_2(t)(\partial\Psi/\partial t)^2 |_{S_3} = 0,\ K_1(t)[\sum_{i=1}^{n}(\partial\Phi/\partial x_i)^2] + K_2(t)(\partial\Phi/\partial t)^2 |_{S_4} = 0$$

where $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$ and $\rho = -\int_0^t \sqrt{-K(\tau)}\, d\tau\ (>0)$ with

$$K(\tau) = K_1(\tau)/K_2(\tau) < 0 \text{ for } K_1(\tau)<0 \text{ and } K_2(\tau)>0 \text{ if } \tau<0.$$

We note that $S_3$ ascends from a fixed point $P(0,t_p), t_p<0$ and intersects the plane $G_0$ in $|x|=1$ for $t<0$, and $S_4$ descends from the fixed origin O(0,0), and intersects $S_3$ for $t<0 : t>t_p$. Furthermore if G is bounded, instead of $S_3$, by a piecewise smooth non-characteristic surface $S_3' : \Theta(x,t)=0$ which intersects the plane $G_0$ in $|x|=1$ and satisfies the non-negative condition

$$K_1(t)[\sum_{i=1}^{n}(\partial\Theta/\partial x_i)^2] + K_2(t)(\partial\Theta/\partial t)^2 |_{S_3'} \geq 0,$$

then the surface $S_3'$ lies inside the characteristic triangle bounded by the characteristic surfaces $S_3$ and $S_4$ of (*).

Assume *the boundary condition*

$$u = 0 \quad \text{on} \quad S_1 \cup S_3 \qquad (**)$$

If $S_3$ is replaced by $S_3'$, the boundary condition (**) is replaced accordingly.

*The Tricomi - Protter problem,* or *Problem* (TP) consists in finding a function $u = u(x,t)$ which satisfies the mixed type partial differential Eq. (*) in G and the boundary condition (**) on the surface portion $S_1 \cup S_3$ of the boundary $\partial G$ of G.

If $S_3$ is replaced by $S_3'$, *the Tricomi - Protter problem,* or *Problem* (TP) is recalled accordingly as *the Frankl - Protter problem,* or *Problem* (FP) .

**Definition 2.1**. A function $u = u(x,t)$ is a *quasi-regular solution* [7-8, 9-10] of Problem (TP) if (i) the Green's theorem (of the integral calculus) is applicable to the integrals

$$\iint_G u\, Lu\, dxdt \ , \ \iint_G u_{x_i} Lu\, dxdt \ \ (i = 1, 2, ..., n) \ \text{and} \ \iint_G u_t\, Lu\, dxdt \ ;$$

(ii) the boundary surface and region integrals, which arise, exist; and (iii) *u* satisfies the mixed type Eq. (*) in G and the boundary condition (**) on $\partial G = S_1 \cup S_3 \cup S_4$.

**Theorem 2.1.** *Consider the parabolic elliptic- hyperbolic Eq.* (*) *and the boundary condition* (**). *Also consider the afore-described bounded simply-connected mixed domain G of the (x, t) - Euclidean space* $\mathbb{R}^{n+1} (n > 2)$. *Besides let us assume that* $K_2(0) > 0$ ; $K_2(t) > 0$ *for all real* $t \neq 0$, *as well as* $K_1(0) = 0$; $K_1(t) > 0$ *for* $t > 0$;*and* $K_1(t) < 0$ *for* $t < 0$.*Furthermore let us assume the conditions:*

$(R_1)$: *The piecewise smooth surface* $S_1 : \chi(x,t) = 0$ *is strongly star-like, such that*

$$\sum_{i=1}^{n} x_i \nu_i + c(t)\nu_{n+1} |_{S_1} \geq 0 \ \ \text{for} \ \ t > 0,$$

*where* $\nu = (\nu_1, \nu_2, ..., \nu_n, \nu_{n+1})$ *is the normal unit vector on* $\partial G = S_1 \cup S_3 \cup S_4$, *and*

$$c(t) = \int_0^t \sqrt{K(\tau)}\, d\tau / \sqrt{K(t)} > 0 \ \ \text{for} \ \ t > 0 \ \text{with} \ K(t) = K_1(t)/K_2(t) > 0,$$

*with characteristic surfaces* $S_3$ and $S_4$ *of (*) described above* ;

$$(R_2): \qquad (2a - nr) - \sum_{i=1}^{n} x_i r_{x_i} - (cr)_t + (K_2 a')' \geq 0 \ \ \text{in G},$$

$$a = [(n-1)/2] + [(K_1(t)K_2(t))' c / 4K_1(t)K_2(t)]$$
$$= [(n-1)/2] + \{[[(K(t))'/K(t)] + 2[(K_2(t))'/K_2(t)]](c/4)\},$$
$$c = \int_0^t \sqrt{|K(\tau)|}\, d\tau / \sqrt{|K(t)|}$$

*in G and* $(\ )' = d(\ )/dt$;

($R_3$): $R(t) = (aK_2\sqrt{-K}\ \rho^{n-1})' + a'K_2\sqrt{-K}\ \rho^{n-1} < 0$ *for* $t < 0$,

$$\rho = -\int_0^t \sqrt{-K(\tau)}\ d\tau\ (>0) \text{ for } t<0;$$

$(R_4)$: $R^*(t) = K_2\ \rho^n - 2(T_{n+1}^2(t)/T'_{n+1}(t) > 0$ for $t < 0$,

*with a real valued function* $T_{n+1} : \mathbb{R} \to \mathbb{R}\ (n \geq 2)$, *such that* $T_{n+1} = T_{n+1}(t) \to 0$ *as* $t \to 0^-$ *and* $T'_{n+1} = dT_{n+1}(t)/dt = -R(t)|_{S_4}\ (>0)$ *,as well as* $f = f(x,t)$ *is continuous in* G, $r = r(x,t)$ *is once-continuously differentiable in* G, $K_i = K_i(t)\ (i=1,2)$ *are monotone increasing continuously differentiable for* $t \in [-k_1, k_2]$ *with* $-k_1 = \inf\{t : (x,t) \in G\}$, $k_2 = \sup\{t : (x,t) \in G\}$, *for* $x = (x_1, x_2, ..., x_n)$ *.Then the Problem* (TP) *has at most one quasi-regular solution in G.*

We note that employing a variation of the $a, b_i\ (i = 1,2,...,n), c$ energy integral method, we obtain the above sufficient conditions $(R_i)\ (i=1,2,3)$ for *the uniqueness of the quasi-regular solutions* of the Tricomi - Protter boundary value problem (*) and (**). If one takes $K_1 = K,\ K_2 = 1$ in this theorem and follows our proof, one obtains a Ph. D. dissertation uniqueness result of the author [7-8].

The following case :
$K_2(0) = 0$; $K_2(t) > 0$ for all real $t \neq 0$, as well as $K_1(0) = 0$; $K_1(t) > 0$ for $t > 0$;and $K_1(t) < 0$ for $t < 0$, where the order of the mixed type equation (*) is degenerated at $t = 0$, is analogous in investigating pertinent quasi-regular solutions of (*).

**Proof .** We apply the well known $a, b_i\ (i = 1,2,...,n), c$ energy integral method with *choices in G:*

$$a = [(n-1)/2] + [(K_1(t)K_2(t))'c/4K_1(t)K_2(t)],\ b_i = x_i,\ c = \int_0^t \sqrt{|K(\tau)|}\,d\tau / \sqrt{|K(t)|} \quad (1)$$

for all $i = 1,2,...,n$.

Then we use the above mixed type Eq. (*) as well as the boundary condition (**).

The above uniqueness method is also useful for proving *the existence of weak solutions* by proving a priori estimates.
First, we assume two quasi-regular solutions $u_1, u_2$ of the Problem (TP).
Then, we claim that $u = u_1 - u_2 = 0$ holds in the domain G. In fact, we investigate

$$0 = J = 2(lu, Lu)_0 = \iint_G 2\, lu\, Lu\, dxdt \quad (2)$$

where

$$lu = a(t)u + \sum_{i=1}^n b_i(x_i)u_{x_i} + c(t)u_t,$$

and $a = a(t),\ b_i = b_i(x_i),\ c = c(t)$ are defined above. Thus

$$Lu = L(u_1 - u_2) = Lu_1 - Lu_2 = f - f = 0 \text{ in G.}$$

We introduce *the new differential identities*

$$2aK_1uu_{x_ix_i} = (2aK_1uu_{x_i})_{x_i} - 2aK_1u_{x_i}^2 - ((aK_1)_{x_i}u^2)_{x_i} + (K_1a)_{x_ix_i}u^2$$
$$= (2aK_1uu_{x_i})_{x_i} - 2aK_1u_{x_i}^2;$$
$$2aK_2uu_{tt} = (2aK_2uu_t)_t - 2aK_2u_t^2 - ((aK_2)'u^2)_t + (K_2a)''u^2,$$
$$2aruu = 2aru^2;$$
$$2aK_2'uu_t = (aK_2'u^2)_t - (aK_2')'u^2;$$

$$2b_iK_1u_{x_i}u_{x_{jj}} = (2b_iK_1u_{x_i}u_{x_j})_{x_j} - (b_iK_1u_{x_j}^2)_{x_i} + (b_iK_1)_{x_i}u_{x_j}^2 - 2(b_iK_1)_{x_j}u_{x_i}u_{x_j}$$
$$= (2b_iK_1u_{x_i}u_{x_j})_{x_j} - (b_iK_1u_{x_j}^2)_{x_i} + (b_iK_1)_{x_i}u_{x_j}^2$$
$$(i \neq j\,;\, i, j := 1, 2, ..., n);$$
$$2b_iK_1u_{x_i}u_{x_ix_i} = (b_iK_1u_{x_i}^2)_{x_i} - (b_iK_1)_{x_i}u_{x_i}^2;$$
$$2b_iK_2u_{x_i}u_{tt} = (2b_iK_2u_{x_i}u_t)_t - (b_iK_2u_t^2)_{x_i} + (b_iK_2)_{x_i}u_t^2 - 2(b_iK_2)_tu_{x_i}u_t;$$
$$2b_iru_{x_i}u = (b_iru^2)_{x_i} - (b_ir)_{x_i}u^2;$$
$$2b_iK_2'u_{x_i}u_t = 2(b_iK_2)_tu_{x_i}u;$$

$$2cK_1u_{x_ix_i}u_t = (2cK_1u_{x_i}u_t)_{x_i} - (cK_1u_{x_i}^2)_t + (cK_1)'u_{x_i}^2 - 2(cK_1)_{x_i}u_{x_i}u_t$$
$$= (2cK_1u_{x_i}u_t)_{x_i} - (cK_1u_{x_i}^2)_t + (cK_1)'u_{x_i}^2;$$
$$2cK_2u_tu_{tt} = (cK_2u_t^2)_t - (cK_2)'u_t^2;$$
$$2cru_tu = (cru^2)_t - (cr)_tu^2;$$
$$2cK_2'u_tu_{tt} = (cK_2'u_t^2)_t - (cK_2')'u_t^2.$$

We note that

$$\sum_{i\neq j}^{n}(b_i\nu_j + b_j\nu_i)u_{x_i}u_{x_j} = 2\sum_{i<j}^{n}(b_i\nu_j + b_j\nu_i)u_{x_i}u_{x_j} = 2\sum_{i\neq j}^{n}(b_i\nu_j)u_{x_i}u_{x_j}.$$

Furthermore we employ the classical Green's theorem of the integral calculus in (2). Therefore if $dV = dxdt$, and dS the surface element ,we get *the fundamental identity*

$$0=J= 2\iint_G [a(t)u + \sum_{i=1}^{n} b_i(x_i)u_{x_i} + c(t)u_t]Lu\, dV$$
$$= \iint_G [2ar - \sum_{i=1}^{n}(b_ir)_{x_i} - (cr)_t + \sum_{i=1}^{n}(K_1a)_{x_ix_i} + (K_2a')']u^2dV$$
$$+ \iint_G \{\sum_{i=1}^{n}[-2aK_1 - (b_iK_1)_{x_i} + \sum_{j\neq i}^{n}(b_jK_1)_{x_j} + (cK_1)']u_{x_i}^2$$
$$+[-2aK_2 + \sum_{i=1}^{n}(b_iK_2)_{x_i} - (cK_2)' + 2cK_2']u_t^2$$
$$-2\sum_{i\neq j}^{n}(K_1b_i)_{x_j}u_{x_i}u_{x_j} - 2\sum_{i=1}^{n}(K_1c)_{x_i}u_{x_i}u_t\}dV$$

$$+ \oint_{\partial G} (\sum_{i=1}^{n} b_i \nu_i + c\nu_{n+1}) r u^2 dS$$

$$+ \oint_{\partial G} \{2au(K_1 \sum_{i=1}^{n} u_{x_i} \nu_i + K_2 u_t \nu_{n+1}) - u^2 [\sum_{i=1}^{n} (K_1 a)_{x_i} \nu_i + a' K_2 \nu_{n+1}]\} dS$$

$$+ \oint_{\partial G} [\sum_{j=1}^{n} (b_j \nu_j - \sum_{i \neq j}^{n} b_i \nu_i - c\nu_{n+1}) K_1 u_{x_j}^2 + (-\sum_{i=1}^{n} b_i \nu_i + c\nu_{n+1}) K_2 u_t^2$$

$$+2 \sum_{i<j}^{n} (b_i \nu_j + b_j \nu_i) K_1 u_{x_i} u_{x_j} + 2 \sum_{i=1}^{n} (b_i K_2 \nu_{n+1} + cK_1 \nu_i) u_{x_i} u_t ] dS$$

$$= I_1 + I_2 + J_1 + J_2 + J_3 . \tag{3}$$

We note that

$$A_i = -2aK_1 - (b_i K_1)_{x_i} + \sum_{j \neq i}^{n} (b_j K_1)_{x_j} + (cK_1)' = 0 \quad \text{for} \quad i = 1, 2, ..., n,$$

$$B = -2aK_2 + \sum_{i=1}^{n} (b_i K_2)_{x_i} - (cK_2)' + 2cK_2' = 0 .$$

Therefore $I_2 = \iint_G (\sum_{i=1}^{n} A_i u_{x_i}^2 + B u_t^2) dV = 0$, because $A_i = B_i = (K_1 b_i)_{x_j} = (K_1 c)_{x_i} = 0.$

Furthermore, from condition $(R_2)$ ,one gets

$$I_1 = \iint_G [(2a - nr) - \sum_{i=1}^{n} x_i r_{x_i} - (cr)_t + (K_2 a')'] u^2 dV \geq 0 .$$

We claim that $J_1 = 0$:

In fact, this is valid from the boundary condition (**) because

$$\nu |_{S_4} = (\nu_1, \nu_2, ..., \nu_n, \nu_{n+1}) = -\nabla \Phi / |\nabla \Phi| = -(x_1, x_2, ..., x_n, N) \sqrt{-K} / N \sqrt{1-K} |_{S_4},$$

such that $N = \rho \sqrt{-K} = -\sqrt{-K} ( \int_0^t \sqrt{-K(\tau)} d\tau ) (> 0)$ for $t < 0$ ,and

$$\nabla \Phi |_{S_4} = (\Phi_{x_1}, \Phi_{x_2}, ..., \Phi_{x_n}, \Phi_t) = (x_1 / \rho, x_2 / \rho, ..., x_n / \rho, \sqrt{-K}), \ |\nabla \Phi| = \sqrt{1 - \mathrm{K}},$$

the characteristic equation $K_1(t) [\sum_{i=1}^{n} (\partial \Phi / \partial x_i)^2] + K_2(t) (\partial \Phi / \partial t)^2 |_{S_4} = 0$ holds, and

$$[\sum_{i=1}^{n} b_i \nu_i + c\nu_{n+1}] |_{S_4} = -(\sqrt{-K} / N \sqrt{1-K}) \{\sum_{i=1}^{n} x_i^2 + [N \int_0^t \sqrt{-K(\tau)} d\tau / \sqrt{-K(t)}]\} |_{S_4} = 0 .$$

We investigate $J_2$:

In fact, let us note that $S_4$:

$$x_1 = \rho \cos\theta \cos\varphi_1 \cos\varphi_2 ... \cos\varphi_{n-3} \cos\varphi_{n-2}; \quad x_2 = \rho \sin\theta \cos\varphi_1 \cos\varphi_2 ... \cos\varphi_{n-3} \cos\varphi_{n-2};$$

$$x_3 = \rho \sin\varphi_1 \cos\varphi_2 ... \cos\varphi_{n-3} \cos\varphi_{n-2}; \quad x_4 = \rho \sin\varphi_2 ... \cos\varphi_{n-3} \cos\varphi_{n-2};$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$x_{n-2} = \rho \sin\varphi_{n-4} \cos\varphi_{n-3} \cos\varphi_{n-2}; \ x_{n-1} = \rho \sin\varphi_{n-3} \cos\varphi_{n-2}; \ x_n = \rho \sin\varphi_{n-2}; \quad t = t,$$

for $\theta \in [0, 2\pi], \varphi_j \in [-\pi/2, \pi/2] \ (j = 1, 2, ..., n-2); x_i \in \mathbb{R}; \mathit{and}\ t \in (-\infty, 0]$.

We consider

$$\overline{u}(t, \theta, \varphi_1, \varphi_2, ..., \varphi_{n-2}) = u(x_1, x_2, ..., x_n, t) |_{S_4}$$

$$= u(\rho\cos\theta\cos\varphi_1...\cos\varphi_{n-2}, \rho\sin\theta\cos\varphi_1...\cos\varphi_{n-2}, ..., \rho\sin\varphi_{n-2}, t).$$

Thus

$$du|_{S_4} = \sum_{i=1}^{n} u_{x_i} dx_i + u_t dt = \overline{u_t} dt + \overline{u_\theta} d\theta + \sum_{j=1}^{n-2} \overline{u_{\varphi_j}} d\varphi_j,$$

such that $\rho = -\int_0^t \sqrt{-K(\tau)}\, d\tau$ yielding $dt = -d\rho(t)/\sqrt{-K(t)}$.

Let us denote $x_i|_{S_4} = f_i(t, \theta, \varphi_1, ..., \varphi_{n-2})$. Thus we obtain

$$dx_i|_{S_4} = (f_i)_t dt + (f_i)_\theta d\theta + \sum_{j=1}^{n-2} (f_j)_{\varphi_j} d\varphi_j \quad \text{and}$$

$$\partial\overline{u}/\partial t = -[\sqrt{-K}(u_{x_1}\cos\theta\cos\varphi_1...\cos\varphi_{n-2} + u_{x_2}\sin\theta\cos\varphi_1...\cos\varphi_{n-2} + ...$$
$$+ u_{x_n}\sin\varphi_{n-2}) - u_t]|_{S_4} = \partial u/\partial T|_{S_4} = \partial\overline{u}/\partial T|_{S_4},$$

which is the derivative in the direction of the tangent vector

$$T = (t_1, t_2, ..., t_n, t_{n+1}) = (\partial x_1/\partial t, \partial x_2/\partial t, ..., \partial x_n/\partial t, \partial t/\partial t)$$
$$= -\sqrt{-K}(x_1, x_2, ..., x_n, -\rho/\sqrt{-K})/\rho,$$

of $u$ in the direction of one of the generators of $S_4$, such that the dot product

$$\nu.T = \sum_{i=1}^{n+1} \nu_i t_i = -[\sum_{i=1}^{n} \Phi_{x_i}(\partial x_i/\partial t) + \Phi_t(\partial t/\partial t)] = 0.$$

Therefore

$$K_1\sum_{i=1}^{n} u_{x_i}\nu_i + K_2 u_t \nu_{n+1}|_{S_4} = K_2\sqrt{-K/(1-K)}[\sqrt{-K}(u_{x_1}\cos\theta\cos\varphi_1...\cos\varphi_{n-2}$$
$$+ ... + u_{x_n}\sin\varphi_{n-2}) - u_t] = -K_2\sqrt{-K/(1-K)}\,(\partial\overline{u}/\partial t) \tag{4}$$

where

$$\cos\theta\cos\varphi_1...\cos\varphi_{n-2} = x_1/\rho = -\sqrt{1-K}\,\nu_1, ..., \sin\varphi_{n-2} = -\sqrt{1-K}\,\nu_n.$$

But $-dx|_{S_4} = \nu_{n+1} dS = -(\Phi_t/|\nabla\Phi|)dS$. Hence

$$dS|_{S_4} = (|\nabla\Phi|/\Phi_t]dx = (\sqrt{1-K}/\sqrt{-K})dx = (\sqrt{1-K}/\sqrt{-K})|J|\, d\varphi d\theta dt,$$

where $dx = \prod_{i=1}^{n} dx_i$, $d\varphi = \prod_{j=1}^{n-2} d\varphi_j$; and

$$J = J(t, \varphi_1, \varphi_2, ..., \varphi_{n-2}) = \partial(x_1, x_2, ..., x_{n-1}, x_n)/\partial(\theta, \varphi_1, ..., \varphi_{n-2}, t)$$
$$= \sqrt{-K}\,(-\int_0^t \sqrt{-K(\tau)}\, d\tau)^{n-1} C(\varphi) = \rho^{n-1}\sqrt{-K}\, C(\varphi),$$

where $C(\varphi) = \prod_{j=1}^{n-2} \cos^j \varphi_j = (\cos\varphi_{n-2})^{n-2}...(\cos\varphi_2)^2 \cos\varphi_1\ (>0)$, for $\varphi_j \in [-\pi/2, \pi/2]\ (j = 1, 2, ..., n-2)$ for $n > 2$ and $J$ the Jacobian.

We note that $C(\varphi) = 1,\ for\ n = 1\ and\ n = 2$. Therefore

$$dS|_{S_4} = \rho^{n-1}\sqrt{1-K}\, C(\varphi) d\varphi d\theta dt. \tag{5}$$

Therefore from (4)-(5), one gets

$$(K_1\sum_{i=1}^{n} u_{x_i}\nu_i + K_2 u_t \nu_{n+1})dS|_{S_4} = -\rho^{n-1} K_2\sqrt{-K}\, C(\varphi)(\partial\overline{u}/\partial t) d\varphi d\theta dt \tag{6}$$

If $F = F(x,t):\mathbb{R}^{n+1} \to \mathbb{R}$ is a given real function, we find

$$J_2^* = \int_{S_4} F(x,t)dS = \int_{S_4'} F(x,t(x))(|\nabla\Phi| / \Phi_t)dx, \tag{7}$$

where $S_4'$ is the projection of $S_4$ into the *x*-space, such that

$|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ ; and $|x| + \int_0^t \sqrt{-K(\tau)}d\tau = 0$ (or $t = t(x)$ ).

Let us denote $\varphi = (\varphi_1, \varphi_2, ..., \varphi_{n-2})$ . Thus from (5) and (7) and denoting

$\Sigma_4 = \{(\theta,\varphi,t) \in \mathbb{R}^{n+1} : 0 \le \theta \le \pi, -\pi/2 \le \varphi_j \le \pi/2 (j = 1,2,...,n-2), t^*(\theta,\varphi) \le t < 0\}$

we get

$$J_2^* = \int_{\Sigma_4} F(-\int_0^t \sqrt{-K(\tau)}d\tau \cos\theta \cos\varphi_1 ... \cos\varphi_{n-2}, ..., t)|J|\sqrt{1-K} / \sqrt{-K}\, dt d\varphi d\theta$$

$$= \int_{\Sigma_4} F(-\int_0^t \sqrt{-K(\tau)}d\tau \cos\theta \cos\varphi_1 ... \cos\varphi_{n-2}, ..., t)\rho^{n-1}(t)C(\varphi)dt d\varphi d\theta$$

We note that $\Sigma_4$ is the region in the $\theta\varphi t - space$ into which the region $S_4'$ is mapped under the following transformation

$$x_1 = \rho\cos\theta\cos\varphi_1\cos\varphi_2 ... \cos\varphi_{n-3}\cos\varphi_{n-2}; \ldots;\ x_n = \rho\sin\varphi_{n-2}. \tag{8}$$

Let us denote

$\Sigma_4' = \{(\theta,\varphi,\rho) \in \mathbb{R}^{n+1} : 0 \le \theta \le \pi, -\pi/2 \le \varphi_j \le \pi/2 (j = 1,2,...,n-2), 0 \le \rho \le R(\theta,\varphi)\}$

the region in the $\theta\varphi\rho - space$ into which either the region $S_4'$ is mapped under the transformation (8) or the region $\Sigma_4$ is mapped under the transformation

$$\rho = -\int_0^t \sqrt{-K(\tau)}\, d\tau . \tag{9}$$

In this latter case the Jacobian is

$$\partial t / \partial\rho = 1 \text{ and } d\rho / dt = -\sqrt{-K(t)} \tag{10}$$

Thus

$$dt d\varphi d\theta = |dt / d\rho| d\rho d\varphi d\theta = (1/\sqrt{-K})d\rho d\varphi d\theta \tag{11}$$

From (5) and the geometry of $S_4$ ,we obtain

$$\nu_{n+1}dS|_{S_4} = (-\sqrt{-K} / \sqrt{1-K})\rho^{n-1}\sqrt{1-K}\, C(\varphi)d\theta d\varphi dt = -\sqrt{-K}\, \rho^{n-1}\, C(\varphi)d\theta d\varphi dt . \tag{12}$$

Therefore from the boundary condition (**), one proves

$$J_2 = -\int_{\Sigma_4} 2\alpha\bar{u}\rho^{n-1}K_2\sqrt{-K}(\partial\bar{u} / \partial t)C(\varphi)d\theta d\varphi dt + \int_{\Sigma_4} \bar{u}^2 a' K_2 \rho^{n-1}\sqrt{-K}C(\varphi)d\theta d\varphi dt . \tag{13}$$

From this and integrating by parts and by virtue of the fact that $\lim_{t\to 0^-} K(t) = 0$ and that $\bar{u}$ vanishes at the upper and lower limits of $S_4$, we get

$$J_2 = \int_{\Sigma_4} (\bar{u})^2[(\alpha K_2\sqrt{-K}\rho^{n-1})' + a' K_2\sqrt{-K}\rho^{n-1}]C(\varphi)d\theta d\varphi dt . \tag{14}$$

We now investigate $J_3$:

In fact, from condition $(R_1)$ and from the boundary condition (**) we prove that

$$J_3 = \int_{S_1 \cup S_3 \cup S_4} Q(u_{x_1}, u_{x_2}, ..., u_{x_n}, u_t) dS = \int_{S_1 \cup S_3} Q(u_{x_1}, u_{x_2}, ..., u_{x_n}, u_t) dS + \int_{S_4} Q(u_{x_1}, u_{x_2}, ..., u_{x_n}, u_t) dS$$

$$= \int_{S_1 \cup S_3} (N^*)^2 [K_1 \sum_{i=1}^n v_i^2 + K_2 v_{n+1}^2][\sum_{i=1}^n x_i v_i + c(t) v_{n+1}] dS + 2\int_{S_4} (\partial \bar{u} / \partial t)^2 (K_2 / \sqrt{1-K}) \rho dS$$

$$\geq 2\int_{S_4} (\partial \bar{u} / \partial t)^2 (K_2 / \sqrt{1-K}) \rho dS$$

$$= 2\int_{\Sigma_4} (\partial \bar{u} / \partial t)^2 (K_2 \rho^n) C(\varphi) d\theta d\varphi dt \tag{15}$$

where $N^*$ is the normalizing factor, such that boundary condition $u|_{S_1} = 0$ implies

$$0 = du|_{S_1 \cup S_3} = \sum_{i=1}^n u_{x_i} du_i + u_t dt; \quad u_{x_i} = N^* v_i \ (i = 1, 2, ..., n), u_t = N^* v_{n+1}, \text{ and}$$

$$[K_1 \sum_{i=1}^n v_i^2 + K_2 v_{n+1}^2]|_{S_1} \geq 0 \quad \text{and characteristic equation } [K_1 \sum_{i=1}^n v_i^2 + K_2 v_{n+1}^2]|_{S_3} = 0 \quad ,$$

as well as the quadratic form Q on the boundary $\partial G = S_1 \cup S_3 \cup S_4$ :

$$Q = Q(u_{x_1}, ..., u_{x_n}, u_t)|_{S_1 \cup S_3 \cup S_4} = \sum_{j=1}^n A_j^* u_{x_j}^2 + B_t^* u_t^2 + \sum_{i<j}^n A_{ij}^* u_{x_i} u_{x_j} + 2\sum_{i=1}^n B_i^* u_{x_i} u_t$$

$$= (N^*)^2 [\sum_{i=1}^n x_i v_i + c(t) v_{n+1}][K_1 \sum_{i=1}^n v_i^2 + K_2 v_{n+1}^2]|_{S_1 \cup S_3} + 2(K_2 / \sqrt{1-K}) \rho (\partial \bar{u} / \partial t)^2 |_{S_4},$$

$$\geq (N^*)^2 [\sum_{i=1}^n x_i v_i + c(t) v_{n+1}][K_1 \sum_{i=1}^n v_i^2 + K_2 v_{n+1}^2]|_{S_1} + 2(K_2 / \sqrt{1-K}) \rho (\partial \bar{u} / \partial t)^2 |_{S_4}$$

$$= 2(K_2 / \sqrt{1-K}) \rho (\partial \bar{u} / \partial t)^2 |_{S_4} \ (\geq 0), \tag{16}$$

where on $S_4$ :

$$A_j^* = \sum_{j=1}^n (b_j v_j - \sum_{i \neq j}^n b_i v_i - c v_{n+1}) K_1 |_{S_4} = -2KK_2 (\sqrt{-K} / N\sqrt{1-K}) x_j^2 \geq 0 (j = 1, 2, ..., n),$$

$$B_t^* = (-\sum_{i=1}^n b_i v_i + c v_{n+1}) K_2 |_{S_4} = 2(K_2 \sqrt{-K} / N\sqrt{1-K}) (\int_0^t \sqrt{-K(\tau)} d\tau)^2 \geq 0,$$

$$A_{ij}^* = \sum_{i<j}^n (b_i v_j + b_j v_i) K_1 |_{S_4} = -2(KK_2 \sqrt{-K} / N\sqrt{1-K}) x_i x_j \ (i < j; i, j := 1, 2, ..., n),$$

$$B_i^* = \sum_{i=1}^n (b_i K_2 v_{n+1} + c K_1 v_i)|_{S_4} = -2(K_2 \sqrt{-K} / \sqrt{1-K}) x_i \ (i = 1, 2, ..., n).$$

Therefore from $J_1 = 0$ and (14)-(15), we obtain

$$J_4 = J_1 + J_2 + J_3 \geq \int_{\Sigma_4} \{(\bar{u})^2 [(aK_2 \sqrt{-K} \rho^{n-1})' + a' K_2 \sqrt{-K} \rho^{n-1}] + 2(\overline{u_t})^2 (K_2 \rho^n)\} C(\varphi) d\theta d\varphi dt \tag{17}$$

where $\overline{u_t} = \partial \bar{u} / \partial t$.

Assuming a real valued function $T_{n+1} : \mathbb{R} \to \mathbb{R} \ (n \geq 2)$, such that $T_{n+1} = T_{n+1}(t) \to 0$ as $t \to 0^-$; $T'_{n+1} = dT_{n+1}(t)/dt = -[(\alpha K_2 \sqrt{-K} \rho^{n-1})' + a' K_2 \sqrt{-K} \rho^{n-1}]|_{S_4} = -R(t)|_{S_4} \ (> 0)$ from condition $(R_3)$. From integration by parts and by virtue of the fact that $\lim_{t \to 0^-} K(t) = 0$ and that $\bar{u}$ vanishes at the upper and lower limits of $S_4$, one gets that

$$\int_{\Sigma_4} T'_{n+1}(t)(\bar{u})^2 C(\varphi)d\theta d\varphi dt = -2\int_{\Sigma_4} T_{n+1}(t)\bar{u}\,\bar{u}_t C(\varphi)dtd\varphi d\theta$$

From this and Cauchy-Schwarz-Buniakowski inequality,as well as by the fact that: $T'_{n+1}(t) > 0$ on $S_4$ $(t < 0)$ and if $T_{n+1}/\sqrt{T'_{n+1}}$ is integrable, one gets on $S_4$ that

$$\int_{\Sigma_4} \mathrm{T}'_{n+1}(t)(\bar{u})^2 C(\varphi)d\theta d\varphi dt = \left| -\int_{\Sigma_4} \mathrm{T}'_{n+1}(t)(\bar{u})^2 C(\varphi)d\theta d\varphi dt \right|$$

$$= 2\left| \int_{\Sigma_4} T_{n+1}(t)\bar{u}\,\bar{u}_t C(\varphi)dtd\varphi d\theta \right| = 2\left| \int [\sqrt{T'_{n+1}(t)}\,\bar{u}][(T_{n+1}(t)/\sqrt{T'_{n+1}(t)})\,\bar{u}_t\,]C(\varphi)dtd\varphi d\theta \right|$$

$$\leq 2[\int_{\Sigma_4} \mathrm{T}'_{n+1}(t)(\bar{u})^2 C(\varphi)\; d\theta d\varphi dt\;]^{1/2}\,[\int_{\Sigma_4} (\mathrm{T}^2_{n+1}(t)/T'_{n+1}(t))(\bar{u}_t)^2 C(\varphi)d\theta d\varphi dt]^{1/2}$$

holds or the inequality

$$\int_{\Sigma_4} \mathrm{T}'_{n+1}(t)(\bar{u})^2 C(\varphi)\; d\theta d\varphi dt \leq 4\int_{\Sigma_4} (\mathrm{T}^2_{n+1}(t)/T'_{n+1}(t))(\bar{u}_t)^2 C(\varphi)d\theta d\varphi dt$$

Therefore from this inequality and (17) we find

$$J_4 = \int_{\Sigma_4} [2K_2\rho^n(\bar{u}_t)^2 - T'_{n+1}(t)(\bar{u})^2]C(\varphi)d\theta d\varphi dt$$

$$\geq 2\int_{\Sigma_4} [\mathrm{K}_2\rho^n - 2(T^2_{n+1}(t)/T'_{n+1}(t)](\bar{u}_t)^2 C(\varphi)d\theta d\varphi dt\;(>0)\text{,as}$$

$$[K_2\;\rho^n - 2(T^2_{n+1}(t)/T'_{n+1}(t)]|_{\Sigma_4} == \{[\mathrm{K}_2\rho^n T'_{n+1}(t) - 2T^2_{n+1}(t)]/T'_{n+1}(t)\}|_{\Sigma_4} > 0\text{, or}$$

$$T'_{n+1}(t)R^*(t) = K_2\rho^n T'_{n+1} - 2T^2_{n+1} > 0 \text{ for } t < 0,$$

which holds by condition $(R_4)$ ,and the proof of our above theorem is complete.

**Corollary 2.1.** *Consider the parabolic elliptic- hyperbolic Eq.* (*) *and the boundary condition* (**). *Also consider the afore-described bounded simply-connected mixed domain G of the (x, t) - Euclidean space* $\mathbb{R}^{n+1}(n > 2)$. *Besides let us assume that* $K_2(0) = 0$; $K_2(t) > 0$ *for all real* $t \neq 0$, *as well as* $K_1(0) = 0$; $K_1(t) > 0$ *for* $t > 0$; *and* $K_1(t) < 0$ *for* $t < 0$.*Furthermore let us assume the conditions:*

$(R_1)$: *The piecewise smooth surface* $S_1 : \chi(x,t) = 0$ *is strongly star-like, such that*

$$\sum_{i=1}^{n} x_i \nu_i + c(t)\nu_{n+1}\,|_{S_1} \geq 0 \text{ for } t > 0,$$

*where* $\nu = (\nu_1, \nu_2, ..., \nu_n, \nu_{n+1})$ *is the normal unit vector on* $\partial G = S_1 \cup S_3 \cup S_4$, *and*

$$c(t) = \int_0^t \sqrt{K(\tau)}\,d\tau / \sqrt{K(t)} > 0 \text{ for } t > 0 \text{ with } K(t) = K_1(t)/K_2(t) > 0,$$

*with* $\lim_{t \to o} K(t) = 0$ *and characteristic surfaces* $S_3$ and $S_4$ *of (*) described above* ;

$(R_j)(j = 2,3,4)$ *hold with a real valued function* $T_{n+1} : \mathbb{R} \to \mathbb{R}$ $(n \geq 2)$, *such that* $T_{n+1} = T_{n+1}(t) \to 0$ *as* $t \to 0^-$ *and* $T'_{n+1} = dT_{n+1}(t)/dt = -R(t)|_{S_4}$ $(> 0)$ *,as well as* $f = f(x,t)$ *is continuous in* G, $r = r(x,t)$ *is once-continuously differentiable in* G, $K_i = K_i(t)$ $(i = 1,2)$ *are monotone increasing continuously differentiable for* $t \in [-k_1, k_2]$ *with* $-k_1 = \inf\{t : (x,t) \in G\}$, $k_2 = \sup\{t : (x,t) \in G\}$; $x = (x_1, x_2, ..., x_n)$.

*Then the Problem* (TP) *has at most one quasi-regular solution in G.*

**Example 2.1.** *Tricomi case:* $K_1 = t; K_2 = 1, such\ that\ K_2(0) > 0$. For $t < 0$:

$K(t) = t;\ \rho = -\int_0^t \sqrt{-K(\tau)}d\tau = -\int_0^t \sqrt{-\tau}d\tau = -2t\sqrt{-t}/3 > 0$;

$T_3 = -t^2/9; T_3' = -2t/9 > 0;\ T_3'(t)R^*(t) = K_2\rho^2 T_3' - 2T_3^2 = 2t^4/27 > 0$;

$T_3^2/T_3' = -t^3/18 \to 0, as\ t \to o^-$.

**Example 2.2.** *Generalized Tricomi case:* $K_1 = t^3; K_2 = t^2, such\ that\ K_2(0) = 0$ .For $t < 0$:

$K(t) = t;\ \rho = -2t\sqrt{-t}/3 > 0$;

$T_3 = -t^4/9; T_3' = -4t^3/9 > 0;\ T_3'(t)R^*(t) = K_2\rho^2 T_3' - 2T_3^2 = 6t^8/81 > 0$;

$T_3^2/T_3' = -t^5/36 \to 0, as\ t \to o^-$.

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# Fundamental Solutions of Two Degenerated Elliptic Equations and Solutions of Boundary Value Problems in Infinite Area

**John Michael Rassias**[a] and **Anvar Hasanov**[b]

[a] National and Capodistrian University of Athens
Pedagogical Department EE
Section of Mathematics and Informatics
Athens, Attikis 15342, Greece.
jrassias@primedu.uoa.gr ; jrassias@tellas.gr

[b]Uzbek Academy of Sciences
Institute of Mathematics
29, F. Hodjaev street, Tashkent 700125, Uzbekistan
anvarhasanov@yahoo.com ; mathinst@uzsci.net

**ABSTRACT**

*In the domain* $D=\{(x,y,z):0<x,-\infty<y<+\infty,-\infty<z<+\infty\}$ *it is considered elliptic type equation with singular coefficient*

$$L_\alpha^\lambda(u)\equiv u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\lambda^2u=0,\ 0<2\alpha<1,\ \lambda=\lambda_1+i\lambda_2,\lambda_1,\lambda_2\in\mathbb{R}.$$

*Fundamental solutions that express through confluent hypergeometric functions of Kummer* $H_3(a,b;c;x,y)$ *from two arguments were found for the given equation. By means of expansion confluent hypergeometric functions of Kummer it is proved, the constructed solutions have a singularity of the order* $1/r$ *at* $r\to 0$. *Further, in case of when* $\lambda^2=-\mu^2$ *for the certain equation by means of found fundamental solutions, boundary value problems are solved in the domain* $D$.

**Keywords:** Singular partial deferential equation, fundamental solutions, hypergeometric Gauss function, confluent hypergeometric function, expansion of hypergeometric function.

**2000 Mathematics Subject Classifications**: primary 35A08

## 1. INTRODUCTION.

In the monograph of R. P. Gilbert [1], applying a method of the complex analysis, integral representation of a solution of generalized bi-axially Helmholtz equation (GBSHE)

$$H_{\alpha,\beta}^\lambda(u)\equiv u_{xx}+u_{yy}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y-\lambda^2u=0,\ \alpha,\beta,\lambda-const, \tag{1.1}$$

via analytical functions is constructed. The obtained formula contains rather bulky series and it is inconvenient in applications. In work [2], applying a method of the complex analysis, inversion formulas and representations are constructed in the explicit form for two important cases of this equation: 1) when $\alpha = 0, \beta > 0$; 2) when $\lambda = 0, \beta > 0$, and the domain, where the equation (1.1) defined, is a circle with the center in an origin of coordinates. Further, two integral equations with cylindrical and Legendre functions in the kernel are solved. There are many scientific works [3-9] in which some problems of the modified forms of the equation (1.1) are studied. For example, in the work [4] a solution of equation (1.1) in the case $\lambda = 0$, expresses by $P_n^{(\alpha,\beta)}(\cos 2\theta)$ - Legendry polynomials. Note, in these works for the considered equations, fundamental solutions were not found.

In this paper we shall consider an elliptic type equation with singular coefficient

$$L_\alpha^\lambda(u) \equiv u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x} u_x + \lambda^2 u = 0,\ 0 < 2\alpha < 1,\ \ \lambda = \lambda_1 + i\lambda_2, \lambda_1, \lambda_2 \in \mathbb{R}. \tag{1.2}$$

in the domain $D = \{(x, y, z): 0 < x, -\infty < y < +\infty, -\infty < z < +\infty\}$. For the given equation fundamental solutions which express through confluent hypergeometric functions of Kummer $H_3(a,b;c;x,y)$ from two arguments were found. By means of expansion confluent hypergeometric functions of Kummer it is proved, the constructed solutions have a singularity of the order $1/r$ at $r \to 0$.

## 2. THE CASE $\lambda = 0$

Let's consider the equation

$$L_\alpha(u) \equiv u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x} u_x = 0,\ 0 < 2\alpha < 1, \tag{2.1}$$

in the field of $D = \{(x, y, z): 0 < x, -\infty < y < +\infty, -\infty < z < +\infty\}$ and a solution of equation (2.1) we search in the form of

$$u = P\omega(\xi), \tag{2.2}$$

where

$$P = (r^2)^{-\alpha - \frac{1}{2}},\quad \left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\} = \left(x \mp x_0\right)^2 + (y - y_0)^2 + (z - z_0)^2,\quad \xi = \frac{r^2 - r_1^2}{r^2}. \tag{2.3}$$

Substituting (2.2) in (2.1), we have

$$A\omega_{\xi\xi} + B\omega_\xi + C\omega = 0, \tag{2.4}$$

where

$$A = P\left[\xi^2{}_x + \xi^2{}_y + \xi^2{}_z\right],$$

$$B = P\xi_{xx} + P\xi_{yy} + P\xi_{zz} + 2P_x\xi_x + 2P_y\xi_y + 2P_z\xi_z + P\frac{2\alpha}{x}\xi_x,$$

$$C = P_{xx} + P_{yy} + P_{zz} + P_x\frac{2\alpha}{x}.$$

After elementary evaluations, we found

$$A = -\frac{4P}{r^2} x^{-1} x_0 \xi(1 - \xi), \tag{2.5}$$

$$B = -\frac{4P}{r^2} x^{-1} x_0 \left[2\alpha - \left(\alpha + \frac{1}{2} + \alpha + 1\right)\xi\right], \tag{2.6}$$

$$C = \frac{4P}{r^2} x^{-1} x_0 \left(\alpha + \frac{1}{2}\right)\alpha. \tag{2.7}$$

Substituting (2.5) - (2.7) equation (2.4), we obtain hypergeometric equation of Gauss [10]

$$\xi\left(1-\xi\right)\omega_{\xi\xi}+\left[2\alpha-\left(\alpha+\frac{1}{2}+\alpha+1\right)\xi\right]\omega_{\xi}-\left(\alpha+\frac{1}{2}\right)\alpha\omega=0. \qquad (2.8)$$

The equation (2.8) has two linearly independent solutions [10]

$$\omega_1\left(x,y,z;x_0,y_0,z_0\right)=F\left(\alpha+\frac{1}{2},\alpha;2\alpha;\xi\right), \qquad (2.9)$$

$$\omega_2\left(x,y,z;x_0,y_0,z_0\right)=\xi^{1-2\alpha}F\left(\frac{3}{2}-\alpha,1-\alpha;2-2\alpha;\xi\right), \qquad (2.10)$$

where

$$F\left(a,b;c;x\right)=\sum_{m=0}^{\infty}\frac{\left(a\right)_m\left(b\right)_m}{\left(c\right)_m m!}x^m,$$

is hypergeometric function of Gauss [10]. Substituting (2.9) - (2.10) in (2.2), we have

$$q_1\left(x,y,z;x_0,y_0,z_0\right)=k_1\left(r^2\right)^{-\alpha-\frac{1}{2}}F\left(\alpha+\frac{1}{2},\alpha;2\alpha;\frac{r^2-r_1^2}{r^2}\right), \qquad (2.11)$$

$$q_2\left(x,y,z;x_0,y_0,z_0\right)=k_2\left(r^2\right)^{\alpha-\frac{3}{2}}x^{1-2\alpha}x_0^{1-2\alpha}F\left(\frac{3}{2}-\alpha,1-\alpha;2-2\alpha;\frac{r^2-r_1^2}{r^2}\right), \qquad (2.12)$$

where $k_1,k_2$ are determined at solving of boundary value problems for the equation (2.1). We shall prove, that the found solutions (2.11) - (2.12) have a singularity $1/r$ at $r\to 0$. Really, by virtue of the formula [10]

$$F\left(a,b;c;x\right)=\left(1-x\right)^{-b}F\left(c-a,b;c;\frac{x}{x-1}\right), \qquad (2.13)$$

the solution (2.11) looks like

$$q_1\left(x,y,z;x_0,y_0,z_0\right)=k_1\left(r^2\right)^{-\frac{1}{2}}\left(r_1^2\right)^{-\alpha}F\left(\alpha-\frac{1}{2},\alpha;2\alpha;\frac{r_1^2-r^2}{r_1^2}\right). \qquad (2.14)$$

Considering value of hypergeometric function of Gauss [10]

$$F\left(a,b;c;1\right)=\frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)},\ c\neq 0,-1,-2,...,\operatorname{Re}c>\operatorname{Re}\left(a+b\right),$$

from equality (2.14) follows that, the solution $q_1\left(x,y,z;x_0,y_0,z_0\right)$ has a singularity $1/r$ at $r\to 0$. It is similarly proved, that the solution (2.12) also has a singularity $1/r$ at $r\to 0$. Thus, the constructed solutions (2.11) - (2.12) are fundamental solutions of equation (2.1).

**3. THE CASE $\lambda\neq 0$**

A solution of the equation (1.2) we search in the form of

$$u=P\omega\left(\xi,\eta\right), \qquad (3.1)$$

where

$$P=\left(r^2\right)^{-\alpha-\frac{1}{2}},\ \ \xi=\frac{r^2-r_1^2}{r^2},\ \eta=\frac{\lambda^2}{4}r^2, \qquad (3.2)$$

$r^2,r_1^2$ are defined from equality (2.3). Substituting (3.1) in the equation (1.2), we have

$$A_1\omega_{\xi\xi}+A_2\omega_{\xi\eta}+A_3\omega_{\eta\eta}+B_1\omega_{\xi}+B_2\omega_{\eta}+C\omega=0. \qquad (3.3)$$

where

$$A_1 = P\xi_x^2 + P\xi_y^2 + P\xi_z^2,$$
$$A_2 = 2P\xi_x\eta_x + 2P\xi_y\eta_y + 2P\xi_z\eta_z,$$
$$A_3 = P\eta_x^2 + P\eta_y^2 + P\eta_z^2,$$
$$B_1 = P\xi_{xx} + P\xi_{yy} + P\xi_{zz} + 2P_x\xi_x + 2P_y\xi_y + 2P_z\xi_z + P\frac{2\alpha}{x}\xi_x,$$
$$B_2 = P\eta_{xx} + P\eta_{yy} + P\eta_{zz} + 2P_x\eta_x + 2P_y\eta_y + 2P_z\eta_z + P\frac{2\alpha}{x}\eta_x,$$
$$C = P_{xx} + P_{yy} + P_{zz} + \frac{2\alpha}{x}P_x + \lambda^2 P.$$

After elementary evaluations, we get

$$A_1 = -\frac{4P}{r^2}x^{-1}x_0\xi\left(1-\xi\right), \tag{3.4}$$
$$A_2 = -\frac{4P}{r^2}x^{-1}x_0\xi\eta - \lambda^2 P\xi, \tag{3.5}$$
$$A_3 = \lambda^2 P\eta, \tag{3.6}$$
$$B_1 = -\frac{4P}{r^2}x^{-1}x_0\left[2\alpha - \left(\alpha + \frac{1}{2} + \alpha + 1\right)\xi\right], \tag{3.7}$$
$$B_2 = -\frac{4P}{r^2}x^{-1}x_0\alpha\eta + \lambda^2 P\left[1 - \left(\alpha + \frac{1}{2}\right)\right], \tag{3.8}$$
$$C = \frac{4P}{r^2}\left(\alpha + \frac{1}{2}\right)\alpha x^{-1}x_0 + \lambda^2 P. \tag{3.9}$$

Substituting the received equalities (3.4) - (3.9) in the equation (3.3), we find system of hypergeometric equation [10]

$$\begin{cases} \xi\left(1-\xi\right)\omega_{\xi\xi} + \xi\eta\omega_{\xi\eta} + \left[2\alpha - \left(\alpha + \frac{1}{2} + \alpha + 1\right)\xi\right]\omega_\xi + \alpha\eta\omega_\eta - \left(\alpha + \frac{1}{2}\right)\alpha\omega = 0 \\ \eta\omega_{\eta\eta} - \xi\omega_{\xi\eta} + \left[1 - \left(\alpha + \frac{1}{2}\right)\right]\omega_\eta + \omega = 0. \end{cases} \tag{3.10}$$

The system of hypergeometric equation (3.10) has two linearly independent solutions

$$\omega_1 = H_3\left(\alpha + \frac{1}{2}, \alpha; 2\alpha; \xi, \eta\right), \tag{3.11}$$
$$\omega_2 = \xi^{1-2\alpha}H_3\left(\frac{3}{2} - \alpha, 1-\alpha; 2-2\alpha; \xi, \eta\right), \tag{3.12}$$

where confluent hypergeometric function of Kummer looks like [10]

$$H_3\left(a,b;c;x,y\right) = \sum_{i,j=0}^{\infty}\frac{\left(a\right)_{i-j}\left(b\right)_i}{\left(c\right)_i i!\,j!}x^i y^j,\ |x|<1,\ \mathrm{Re}\,c > \mathrm{Re}\,b > 0. \tag{3.13}$$

Substituting solutions (3.11) - (3.12), in (3.1), we find solutions of the equation (1.2)

$$q_1\left(x,y,z;x_0,y_0,z_0\right) = k_1\left(r^2\right)^{-\alpha-\frac{1}{2}}H_3\left(\alpha + \frac{1}{2}, \alpha; 2\alpha; \frac{r^2 - r_1^2}{r^2}, \frac{\lambda^2}{4}r^2\right), \tag{3.14}$$
$$q_2\left(x,y,z;x_0,y_0,z_0\right) = k_2\left(r^2\right)^{-\alpha-\frac{1}{2}}\xi^{1-2\alpha}H_3\left(\frac{3}{2} - \alpha, 1-\alpha; 2-2\alpha; \frac{r^2 - r_1^2}{r^2}, \frac{\lambda^2}{4}r^2\right), \tag{3.15}$$

where $k_1, k_2$, and they are defined at solving of boundary value problems for the equation (1.2).

Note that, for confluent hypergeometric function of Kummer $H_3(a,b;c;x,y)$ takes place the following formulas of expansion and integral representation

$$H_3(a,b;c;x,y) = \sum_{i=0}^{\infty} \frac{(b)_i}{(1-a)_i (c)_i\, i!} x^i y^i F(a,b+i;c+i;x)\, {}_0F_1(1-a+i;-y), \tag{3.16}$$

$$H_3(a,b;c;x,y) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(1-a)_i\, i!} y^i F(a-i,b;c;x), \tag{3.17}$$

$$H_3(a,b;c;x,y) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i\, i!} x^i \overline{J}_{-a-i}\left[2\sqrt{y}\right], \tag{3.18}$$

$$H_3(a,b;c;x,y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, \overline{J}_{-a}\left[2\sqrt{y(1-xt)}\right] dt, \tag{3.19}$$

$\overline{J}_\nu\left[2\sqrt{z}\right] = \Gamma(1+\nu) z^{-\frac{\nu}{2}} J_\nu\left[2\sqrt{z}\right] = {}_0F_1(1+\nu;-z)$ is function Bessel-Clifford [10]. The proof of expansion (3.16) are based on symbolical method of Burchnall-Chaundy [11, 12]. We shall note, that in work [13] for hypergeometric function of Lauricella from many variables formulas of expansion were found.

Let's prove that, the received particular solutions (3.14) - (3.15) have a singularity $1/r$ at $r \to 0$. We shall consider a particular solution (3.14). By virtue of expansion (3.17) and considering the formula (2.13), (3.14) we shall transform to an aspect

$$\begin{aligned} q_1(x,y,z;x_0,y_0,z_0) &= k_1 \left(r^2\right)^{-\frac{1}{2}} \left(r_1^2\right)^{-\alpha} F\left(\alpha - \frac{1}{2}, \alpha; 2\alpha; \frac{r_1^2 - r^2}{r_1^2}\right) + \\ &+ k_1 \left(r_1^2\right)^{-\alpha-\frac{1}{2}} \sum_{i=1}^{\infty} \frac{(-1)^i}{\left(\frac{1}{2}-\alpha\right)_i i!} \left(\frac{\lambda^2}{4} r_1^2\right)^i F\left(\alpha + \frac{1}{2} - i, \alpha; 2\alpha; \frac{r_1^2 - r^2}{r_1^2}\right). \end{aligned} \tag{3.20}$$

From identity (3.20) follows, that the first composed converts infinity of the order $1/r$, and the second composed is limited at $r \to 0$.

If in the solutions (3.14) and (3.15) $\lambda = 0$ they coincide to the solutions (2.11) and (2.12) of the equation (2.1).

## 4. THE STATEMENT AND UNIQUENESS OF THE SOLUTION OF BOUNDARY VALUE PROBLEMS

We consider the equation

$$L_\alpha^\lambda(u) \equiv u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x} u_x - \mu^2 u = 0,\ 0 < 2\alpha < 1,\ \ \mu \in \mathbb{R}, \tag{4.1}$$

in the domain $D = \{(x,y,z) : 0 < x, -\infty < y < +\infty, -\infty < z < +\infty\}$.

**Problem** $D^\infty$. Find a solution of the equation (4.1) from the class $C(\overline{D}) \cap C^2(D)$, satisfying the following conditions

$$u\left(x,y,z\right)\Big|_{x=0}=\tau\left(y,z\right),\ -\infty<y<+\infty,\ -\infty<z<+\infty, \tag{4.2}$$

$$\lim_{x\to\infty}u\left(x,y,z\right)=0. \tag{4.3}$$

$\tau\left(y,z\right)$ is a continuous and bounded function in a plane $-\infty<y<+\infty,\ -\infty<z<+\infty.$

**Problem** $N^{\infty}$. Find a solution of the equation (4.1) from the class $C\left(\bar{D}\right)\cap C^{2}\left(D\right)$, satisfying conditions

$$x^{2\alpha}u_{x}\left(x,y,z\right)\Big|_{x=0}=\nu\left(y,z\right),\ -\infty<y<+\infty,\ -\infty<z<+\infty, \tag{4.4}$$

$$\lim_{x\to\infty}u\left(x,y,z\right)=0. \tag{4.5}$$

$\nu\left(y,z\right)$ is a continuous function in a plane $-\infty<y<+\infty,\ -\infty<z<+\infty.$

**Theorem.** If for enough big $R$ the following conditions are satisfied

$$\left|u\left(x,y,z\right)\right|\le\frac{c_{1}}{R},\ \left|x^{2\alpha}u_{x}\left(x,y,z\right)\right|\le\frac{c_{2}}{R^{1+\varepsilon}},\ \varepsilon>0, \tag{4.6}$$

where $c_1,c_2$ are const, $R^2=x^2+y^2+z^2$, then solutions of a problem $D^{\infty}$, $N^{\infty}$ for the equation (4.1), are unique.

At the proof of the theorem will be used the following identities

$$x^{2\alpha}\left[u_{x}^{2}+u_{y}^{2}+u_{z}^{2}+\mu^{2}u^{2}\right]\equiv\frac{\partial}{\partial x}\left(x^{2\alpha}uu_{x}\right)+\frac{\partial}{\partial y}\left(x^{2\alpha}uu_{y}\right)+\frac{\partial}{\partial z}\left(x^{2\alpha}uu_{z}\right), \tag{4.7}$$

and Green's formula.

## 5. EXISTENCE OF A SOLUTION OF A PROBLEM $D^{\infty}$, $N^{\infty}$

Using a fundamental solution $q_2\left(x,y,z;x_0,y_0,z_0\right)$ of the equation (4.1), we consider the formula

$$u\left(x_0,y_0,z_0\right)=\int_0^{\infty}\int_0^{\infty}x^{2\alpha}\tau\left(y,z\right)\frac{\partial}{\partial x}q_2\left(0,y,z;x_0,y_0,z_0\right)\Big|_{x=0}dydz, \tag{5.1}$$

where $\left(x_0,y_0,z_0\right)\in D$. The formula (5.1) is a solution of a problem $D^{\infty}$ and satisfies the equation (4.1). By virtue of (3.15) from (5.1) considering

$$H_3\left(a,b;c;0,y\right)={}_0F_1\left(1-a;-y\right)=\sum_{m=0}^{\infty}\frac{\left(-y\right)^m}{\left(1-a\right)_m m!}$$

we found

$$u\left(x_0,y_0,z_0\right)$$

$$=k_2\left(1-2\alpha\right)x_0^{1-2\alpha}\int_0^{\infty}\int_0^{\infty}\tau\left(y,z\right)\frac{{}_0F_1\left(\alpha-\frac{1}{2};-\frac{\mu^2}{4}\left[x_0^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2\right]\right)}{\left[x_0^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2\right]^{\frac{3}{2}-\alpha}}dydz. \tag{5.2}$$

It is easy to see, that the solution (5.2) to converge absolutely.

We prove, that function $u\left(x_0,y_0,z_0\right)$ is limited and $u\left(x_0,y_0,z_0\right)\to\tau\left(y_0,z_0\right)$ at $x_0\to0$. Really, having made a change of variables, $y=y_0+x_0t,\ z=z_0+x_0s$ we shall transform the formula (5.2) to an aspect

$$u\left(x_0,y_0,z_0\right) = k_2\left(1-2\alpha\right)\int_{-\frac{y_0}{x_0}}^{\infty}\int_{-\frac{z_0}{x_0}}^{\infty}\tau\left(y_0+x_0t,z_0+x_0s\right)\frac{{}_0F_1\left(\alpha-\frac{1}{2};-\frac{\mu^2}{4}\left[x_0^2\left(1+t^2+s^2\right)\right]\right)}{\left(1+t^2+s^2\right)^{\frac{3}{2}-\alpha}}dtds. \tag{5.3}$$

At $x_0 \to 0$, we have

$$\lim_{x_0\to 0} u\left(x_0,y_0,z_0\right) = k_2\left(1-2\alpha\right)\tau\left(y_0,z_0\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dtds}{\left(1+t^2+s^2\right)^{\frac{3}{2}-\alpha}}. \tag{5.4}$$

Value of last integral is known

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dtds}{\left(1+t^2+s^2\right)^{\frac{3}{2}-\alpha}} = \frac{2\pi}{\left(1-2\alpha\right)}. \tag{5.5}$$

Considering value

$$k_2 = \frac{1}{2\pi}, \tag{5.6}$$

from (5.3) we find, that $\left|u\left(x_0,y_0,z_0\right)\right| \le \max\left|\tau\left(y_0,z_0\right)\right| = M,$ hence $u\left(x_0,y_0,z_0\right)$ it is limited in the field of $D$. From (5.3) we get

$$\left|u\left(x_0,y_0,z_0\right)-\tau\left(y_0,z_0\right)\right| \le k_2\left(1-2\alpha\right) \cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\tau\left(y_0+x_0t,z_0+x_0s\right)-\tau\left(y_0,z_0\right)\right|\frac{{}_0F_1\left(\alpha-\frac{1}{2};-\frac{\mu^2}{4}\left[x_0^2\left(1+t^2+s^2\right)\right]\right)}{\left(1+t^2+s^2\right)^{\frac{3}{2}-\alpha}}dtds. \tag{5.7}$$

By virtue of limitation of the function $\tau\left(y_0,z_0\right)$ at any $x_0, y_0, z_0, t, s$

$$\left|\tau\left(y_0+x_0t,z_0+x_0s\right)-\tau\left(y_0,z_0\right)\right| \le 2M.$$

Let $\varepsilon > 0$ any positive number. Then there is such so big positive number $N,$ that

$$2Mk_2\left(1-2\alpha\right)\int_{-\infty}^{-N}\int_{-\infty}^{-N}\frac{{}_0F_1\left(\alpha-\frac{1}{2};-\frac{\mu^2}{4}\left[x_0^2\left(1+t^2+s^2\right)\right]\right)}{\left(1+t^2+s^2\right)^{\frac{3}{2}-\alpha}}dtds < \frac{\varepsilon}{3}, \tag{5.8}$$

$$2Mk_2\left(1-2\alpha\right)\int_{N}^{\infty}\int_{N}^{\infty}\frac{{}_0F_1\left(\alpha-\frac{1}{2};-\frac{\mu^2}{4}\left[x_0^2\left(1+t^2+s^2\right)\right]\right)}{\left(1+t^2+s^2\right)^{\frac{3}{2}-\alpha}}dtds < \frac{\varepsilon}{3}. \tag{5.9}$$

Then from (5.7) will follow

$$\left|u\left(x_0,y_0,z_0\right)-\tau\left(y_0,z_0\right)\right| \le \frac{2}{3}\varepsilon + k_2\left(1-2\alpha\right) \cdot\int_{-N}^{N}\int_{-N}^{N}\left|\tau\left(y_0+x_0t,z_0+x_0s\right)-\tau\left(y_0,z_0\right)\right|\frac{{}_0F_1\left(\alpha-\frac{1}{2};-\frac{\mu^2}{4}\left[x_0^2\left(1+t^2+s^2\right)\right]\right)}{\left(1+t^2+s^2\right)^{\frac{3}{2}-\alpha}}dtds. \tag{5.10}$$

By virtue of a continuity $\tau(y_0, z_0)$ it is possible to conclude, that at $x_0$ enough close to zero, and at $|t| \le N, |s| \le N$ we have

$$\left|\tau(y_0 + x_0 t, z_0 + x_0 s) - \tau(y_0, z_0)\right| < \frac{\varepsilon}{3},$$

and in view of an inequality (5.8), (5.9) it is defined

$$\left|u(x_0, y_0, z_0) - \tau(y_0, z_0)\right|$$

$$< \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon k_2 (1 - 2\alpha) \int_{-N}^{N}\int_{-N}^{N} \frac{{}_0F_1\left(\alpha - \frac{1}{2}; -\frac{\mu^2}{4}\left[x_0^2\left(1 + t^2 + s^2\right)\right]\right)}{\left(1 + t^2 + s^2\right)^{\frac{3}{2} - \alpha}} dtds. \quad (5.11)$$

By virtue of that

$$\lim_{x_0 \to 0} {}_0F_1\left(\alpha - \frac{1}{2}; -\frac{\mu^2}{4}\left[x_0^2\left(1 + t^2 + s^2\right)\right]\right) = 1,$$

From (5.11) in view of value of an integral (5.5) we have $\left|u(x_0, y_0, z_0) - \tau(y_0, z_0)\right| < \varepsilon$. Thus takes place equalities $\lim_{x_0 \to 0} u(x_0, y_0, z_0) = \tau(y_0, z_0)$.

Similarly, using a fundamental solution $q_1(x, y, z; x_0, y_0, z_0)$ of the equation (4.1), we shall receive a solution of a problem $N^\infty$. This solution looks like

$$u(x_0, y_0, z_0) = -k_1 \int_0^\infty\int_0^\infty \nu(y, z) \frac{{}_0F_1\left(\frac{1}{2} - \alpha; -\frac{\mu^2}{4}\left[x_0^2 + (y - y_0)^2 + (z - z_0)^2\right]\right)}{\left[x_0^2 + (y - y_0)^2 + (z - z_0)^2\right]^{\alpha + \frac{1}{2}}} dydz, \quad (5.12)$$

where

$$k_1 = \frac{1}{2\pi}.$$

Also as in the case of a problem $D^\infty$ it is possible to be convinced, that the solution (5.12) satisfies to conditions of a problem $N^\infty$.

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# Solvability of the oblique derivative problem for second order equations of mixed type with nonsmooth degenerate curve

**J. M. Rassias**[1] **and Guochun Wen**[2]

[1]Pedagogical Department,
University of Athens,
Attikis 15342, Greece
e-mail: jrassias@tellas.gr

[2]School of Mathematical Sciences,
Peking University,
Beijing 100871, China
e-mail: wengc@math.pku.edu.cn

**ABSTRACT**

*In [1]-[7], the authors posed and discussed the Tricomi problem of second order equations of mixed type, but they only consider some special mixed equations. In [4], the author discussed the uniqueness of solutions of the Tricomi problem for a second order mixed equation with nonsmooth degenerate line. The present article deals with the oblique derivative problem for general second order mixed equations with nonsmooth parabolic degenerate line, which includes the Tricomi problem as a special case, we first give the formulation of the above problem, and then prove the solvability of the problem for the mixed equations with nonsmooth degenerate line. Here we introduce the notation (2.1) below, such that the second order equation of mixed type can be reduced to the mixed complex equation of first order and then we can use the advantage of complex analytic method, otherwise the complex analytic method cannot be used.*

**Keywords:** Oblique derivative problem, mixed equations, nonsmooth degenerate line.

**2000 Mathematics Subject Classification:** 35N99.

## 1 Formulation of the oblique derivative problem for degenerate mixed equations with nonsmooth degenerate line

Let *D* be a simply connected bounded domain in the complex plane $\mathbf{C}$ with the boundary $\partial D = \Gamma \cup L$, where $\Gamma(\subset \{x > 0, y > 0\}) \in C^2_\alpha(0 < \alpha < 1)$ is a curve with the end points $z = 1, i$, and $L = L_1 \cup L_2 \cup L_3 \cup L_4$, where $L_1, L_2, L_3, L_4$ are four characteristic lines with the slopes $-H_2(x)/H_1(y), H_2(x)/H_1(y), -H_2(x)/H_1(y), H_2(x)/H_1(y)$ passing through the points

$z = x + jy = 0, 1, 0, j$ respectively as follows

$$\begin{aligned} &L_1=\{-G_1(y)=-\int_0^y H_1(t)dt=G_2(x)=\int_0^x H_2(t)dt, x\in(0,x_1)\},\\ &L_2=\{-G_1(y)=-\int_0^y H_1(t)dt=\int_x^1 H_2(t)dt=G_2(1)-G_2(x), x\in(x_1,1)\},\\ &L_3=\{G_1(y)=\int_0^y H_1(t)dt=-\int_0^x H_2(t)dt=-G_2(x), y\in(0,y_2)\},\\ &L_4=\{G_1(1)-G_1(y)=\int_y^1 H_1(t)dt=-\int_0^x H_2(t)dt=-G_2(x), y\in(y_2,1)\}, \end{aligned} \tag{1.1}$$

in which $H_1(y)=\sqrt{|K_1(y)|}, H_2(x)=\sqrt{|K_2(x)|}$, $K_1(0)=0$, $K_2(0)=0$, $K_1(y)=\text{sgn}y|y|^{m_1}h_1(y)$, $K_2(x)=\text{sgn}x|x|^{m_2}h_2(x)$, $m_1, m_2$ are positive constants, $h_1(y), h_2(x)$ in $\overline{D}$ are continuously differentiable positive functions, and $(x_1, y_1)$, $(x_2, y_2)$ are the intersection points of $L_1, L_2$ and $L_3, L_4$ respectively. There is no harm in assuming that the boundary $\Gamma$ of the domain $D$ is a smooth curve including the line segments $\text{Re}z = 1$ and $\text{Im}z = 1$ near the points $z = 1$ and $i$ respectively, otherwise by using a conformal mapping, the requirement can be realized. Denote $D^+ = D\cap\{x > 0, y > 0\}$, $D^- = D_1^-\cup D_2^-, D_1^- = D\cap\{y < 0\}, D_2^- = D\cap\{x < 0\}$. In this article we use the notation of the complex number in $D^+$ and the hyperbolic number in $D^-$ (see [10]). Now we introduce the second order linear equation of mixed type with nonsmooth degenerate line

$$Lu=K_1(y)u_{xx}+K_2(x)u_{yy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u=-d(x,y) \text{ in } \overline{D}, \tag{1.2}$$

especially the above equation with $K_2(x) = 1, a = b = c = d = 0$ is the famous Chaplygin equation in gas dynamics. Suppose that the coefficients of (1.2) satisfy **Condition** $C$, namely the real functions $a, b, c, d$ are measurable in $z \in D^+$ and continuous in $\overline{D^-}$, and satisfy

$$\begin{gathered} L_\infty[\eta, D^+],\ \eta = a, b, c,\ L_\infty[d, \overline{D^+}] \le k_1,\ c \le 0 \text{ in } D^+,\\ \eta|x|^{-m_2/2} = O(1) \text{ as } z = x + iy \to 0,\ \eta = a, b, c, d,\\ |a|/H_1 = \varepsilon_1(y)k_0 \text{ in } \overline{D_1^-}, m_1 \ge 2,\ |b|/H_2 = \varepsilon_2(x)k_0 \text{ in } \overline{D_2^-}, m_2 \ge 2,\\ \tilde{C}[\eta,\overline{D_1^-}]=C[\eta,\overline{D_1^-}]+C[\eta_x,\overline{D_1^-}]\le k_0, \eta=a,b,c, \tilde{C}[d,\overline{D_1^-}]\le k_1,\\ \hat{C}[\eta,\overline{D_2^-}]=C[\eta,\overline{D_2^-}]+C[\eta_y,\overline{D_2^-}]\le k_0, \eta=a,b,c, \hat{C}[d,\overline{D_2^-}]\le k_1, \end{gathered} \tag{1.3}$$

in which, $k_0(\ge \max_{l=1,2}[2\sqrt{h_l}, 1/\sqrt{h_l}])$, $k_1(\ge \max[1, 6k_0])$ are positive constants, $\varepsilon_1(y) \to 0$ as $y \to 0$ and $\varepsilon_2(x) \to 0$ as $x \to 0$. If $H_1(y) = [|y|^{m_1}h_1(y)]^{1/2}$, $H_2(x) = [|x|^{m_2}h_2(x)]^{1/2}$, here $m_1, m_2$ are positive numbers, then

$$\begin{aligned} \pm Y = \pm G_1(y) = \pm\int_0^y H_1(t)dt,\ |Y| \le \frac{k_0}{m_1+2}|y|^{(m_1+2)/2},\\ \pm X = \pm G_2(x) = \pm\int_0^x H_2(t)dt,\ |X| \le \frac{k_0}{m_2+2}|x|^{(m_2+2)/2}, \end{aligned} \tag{1.4}$$

and their inverse functions $y = \pm|(\pm G_1)^{-1}(\pm Y)|$, $x = \pm|(\pm G_2)^{-1}(\pm X)|$ satisfy the inequalities

$$\begin{aligned} |y|=|(\pm G_1)^{-1}(\pm Y)|\le \Big(\frac{k_0(m_1+2)}{2}\Big)^{2/(m_1+2)}|Y|^{2/(m_1+2)}=J_1|Y|^{2/(m_1+2)},\\ |x|=|(\pm G_2)^{-1}(\pm X)|\le \Big(\frac{k_0(m_2+2)}{2}\Big)^{2/(m_2+2)}|X|^{2/(m_2+2)}=J_2|X|^{2/(m_2+2)}. \end{aligned} \tag{1.5}$$

The oblique derivative boundary value problem for equation (1.1) may be formulated as follows:

**Problem** $P$ Find a continuous solution $u(z)$ of (1.2) in $\overline{D}$, where $u_x, u_y$ are continuous in $D^* = \overline{D}\backslash\{1, i, 0\}$, and satisfy the boundary conditions

$$\begin{gathered}\frac{1}{2}\frac{\partial u}{\partial l} = \frac{1}{H(z)}\mathrm{Re}[\overline{\lambda(z)}u_{\tilde{z}}] = \mathrm{Re}[\overline{\Lambda(z)}u_z] = r(z) \ \text{ on } \Gamma\cup L_2\cup L_4,\\ \frac{1}{H(z)}\mathrm{Im}[\overline{\lambda(z)}u_{\tilde{z}}]|_{z=z_l} = \mathrm{Im}[\overline{\Lambda(z)}u_z]|_{z=z_l} = b_l, l=1,2, u(1)=b_0, u(i)=b_3,\end{gathered} \tag{1.6}$$

in which $l$ is a given vector at every point $z \in \Gamma \cup L_2 \cup L_4$, $H(z) = H_1(y)$ or $H_2(x)$, $u_{\tilde{z}} = [H_1(y)u_x - iH_2(x)u_y]/2$, $\Lambda(z) = \cos(l, x) - i\cos(l, y)$ and $\lambda(z) = \mathrm{Re}\lambda(z) + i\mathrm{Im}\lambda(z)$, if $z \in \Gamma \cup L_2 \cup L_4$, $b_l(l = 0, 1, 2, 3)$ are real constants, and $r(z), b_l(l = 0, 1, 2, 3)$ satisfy the conditions

$$\begin{gathered}C^1_\alpha[\lambda(z), \Gamma] \le k_0, C^1_\alpha[\lambda(z), L_2\cup L_4] \le k_0, C^1_\alpha[r(z), \Gamma] \le k_2,\\ C^1_\alpha[r(z), L_2\cup L_4] \le k_2, \ \cos(l, n) \ge 0 \text{ on } \Gamma\cup L_2\cup L_4,\\ |b_l| \le k_2, l=0,1,2,3, \max_{z\in L_2}\frac{1}{|a(z)+b(z)|} \le k_0, \max_{z\in L_4}\frac{1}{|a(z)+b(z)|} \le k_0,\end{gathered} \tag{1.7}$$

in which $n$ is the outward normal vector at every point on $\Gamma$, $\alpha\,(0 < \alpha < 1), k_0, k_2$ are non-negative constants. The number

$$K = \frac{1}{2}(K_1 + K_2 + K_3) \tag{1.8}$$

is called the index of Problem $P$, where

$$K_j = \left[\frac{\phi_j}{\pi}\right]^{\circ} + J_j, J_j = 0 \text{ or } 1, e^{i\phi_j} = \frac{\lambda(t_j-0)}{\lambda(t_j+0)}, \gamma_j = \frac{\phi_j}{\pi} - K_j, j=1,2,3, \tag{1.9}$$

in which $t_1 = 1, t_2 = i, t_3 = 0$. Here $K = 0$ or $1/2$ on the boundary $\partial D$ of $D$ is chosen, because otherwise it is sufficient to multiply the complex equation (2.2) below by the function $X_0(z) = z$ or $X_0(z) = z(z + 1/2)(z - 1/2)$, then the index of $\hat{\lambda}(z)$ on $\partial D$ is equal to $\hat{K} = 0$ or $1/2$.

Moreover the Tricomi problem for equation (1.1) may be formulated as follows:

**Problem** $T$ Find a continuous solution $u(z)$ of (1.1) in $\overline{D}$, where $u_x, u_y$ are continuous in $D^* = \overline{D}\backslash\{1, i, 0\}$, and satisfy the boundary conditions

$$u(z) = \phi(z) \text{ on } \Gamma, \ u(z) = \psi_1(x) \text{ on } L_2, u(z) = \psi_2(y) \text{ on } L_4, \tag{1.10}$$

where $\phi(1) = \psi_1(1), \phi(i) = \psi_2(i)$, and $\phi(z), \psi_1(x), \psi_2(y)$ satisfy the conditions

$$C^2_\alpha[\phi(z), \Gamma] \le k_2, \ C^2_\alpha[\psi_1(x), L_2] \le k_2, \ C^2_\alpha[\psi_2(y), L_4] \le k_2, \tag{1.11}$$

in which $\alpha\,(0<\alpha<1), k_2$ are positive constants.

We find the derivative for (1.10) according to the parameter $s = \mathrm{Im}\, z = y$ on $\Gamma$ near $z = 1$ and the parameter $s = \mathrm{Re}\, z = x$ on $\Gamma$ near $z = i$, and obtain

$$\begin{gathered}\mathrm{Re}[\overline{\lambda(z)}(U + iV)] = \mathrm{Re}[\overline{\lambda(z)}(H_1(y)u_x - iH_2(x)u_y)]/2 = R(z) \ \text{ on } \Gamma,\\ \mathrm{Re}[\overline{\lambda(z)}(U + jV)] = \mathrm{Re}[\overline{\lambda(z)}(H_1(y)u_x - jH_2(x)u_y)]/2 = R(z) \text{ on } L_2 \cup L_4,\\ \mathrm{Im}[\overline{\lambda(z)}(U+jV)]_{z=z_1} = \mathrm{Im}[\overline{\lambda(z)}(H_1(y)u_x - jH_2(x)u_y)]/2|_{z=z_1} = b_1,\\ \mathrm{Im}[\overline{\lambda(z)}(U+jV)]_{z=z_2} = \mathrm{Im}[\overline{\lambda(z)}(H_1(y)u_x - jH_2(x)u_y)]/2|_{z=z_2} = b_2,\end{gathered}$$

where

$$U(z)=\frac{1}{2}u_x, V(z)=-\frac{1}{2}v_y, b_0=\phi(1), b_1=\frac{1}{2\sqrt{2}}[-H_1(y_1)\psi_1'(x_1)], b_2=\frac{1}{2\sqrt{2}}[-H_2(x_2)\psi_2'(y_2)],$$

and

$$\lambda(z)=\begin{cases} i, \\ 1, \\ (1+j)/\sqrt{2}, \\ (1+j)/\sqrt{2}, \\ j, \\ 1, \end{cases} R(z)=\begin{cases} H_2(x)\phi'(y)/2 \text{ on } \Gamma \text{ at } z=1, \\ H_1(y)\phi'(x)/2 \text{ on } \Gamma \text{ at } z=i, \\ H_1(y)\psi_1'(x)/2\sqrt{2} \text{ on } L_2, \\ H_2(x)\psi_2'(y)/2\sqrt{2} \text{ on } L_4, \\ -\tilde{R}_0(x) \text{ on } L_0'=\overline{D^-}\cap\{y=0\}, \\ \hat{R}_0(y) \text{ on } L_0''=\overline{D^-}\cap\{x=0\}. \end{cases}$$

where $R_0(x)$ on $L_0'$ and $\hat{R}_0(y)$ on $L_0''$ are undermined functions, $L_0 = L_0' \cup L_0''$. Denoting $t_1 = 1$, $t_2 = i$, $t_3 = 0$, we have

$$e^{i\phi_1} = \frac{\lambda(t_1-0)}{\lambda(t_1+0)} = e^{\pi i/2-\pi i/2} = e^{0\pi i}, \gamma_1 = 0 - K_1 = 0, K_1 = 0,$$

$$e^{i\phi_2} = \frac{\lambda(t_2-0)}{\lambda(t_2+0)} = e^{0\pi i-0\pi i} = e^{0\pi i}, \gamma_2 = 0 - K_2 = 0,\ K_2 = 0,$$

$$e^{i\phi_3}=\frac{\lambda(t_3-0)}{\lambda(t_3+0)}=e^{0\pi i-\pi i/2}=e^{-\pi i/2}, \gamma_3=\frac{-\pi/2}{\pi}-K_3=-\frac{1}{2}, K_3=0,$$

hence the index of $\lambda(z)$ on $\Gamma \cup L_0' \cup L_0''$ is $K = (K_1 + K_2 + K_3)/2 = 0$. Obviously the Tricomi problem for Chaplygin equation is a special case of Problem $P$ for equation (1.1). Because the this case we have $\cos(l, n) \equiv 0$ on $\Gamma$, from the boundary condition (1.6), we can determine the value $u(i)$ by the value $u(1)$, namely

$$u(i)=2\mathrm{Re}\int_i^1 u_z dz + b_0=2\int_0^S \mathrm{Re}[z'(s)u_z]ds+b_0=2\int_0^S r(z)ds + b_0, \tag{1.12}$$

in which $\overline{\Lambda(z)} = z'(s)$ on $\Gamma$, $z(s)$ is a parameter expression of arc length $s$ of $\Gamma$ with the condition $z(0) = 0$, and $S$ is the length of the curve $\Gamma$, the condition $u(i) = b_3$ is not necessary and can be cancelled. If $H_2(x)u_y(x) = -2\tilde{R}_0(x)$ on $L_0'$, $H_1(y)u_x(y) = 2\hat{R}_0(y)$ on $L_0''$ are replaced by $\mathrm{Re}[\overline{\lambda(z)}W(z)] = H_1(y)u_x/2 = 0, \lambda(z) = 1$ on $L_0'$ and $\mathrm{Re}[\overline{\lambda(z)}W(z)] = -H_2(x)u_y/2 = 0, \lambda(z) = i$ on $L_0''$ respectively, then similarly we can choose the corresponding numbers $\gamma_1 = \gamma_2 = -1/2$, $\gamma_3 = 1/2$, and the index $K = 0$ of the function $\lambda(z)$ on $\partial D^+$, we can add one point condition $u(0) = 0$.

Noting that $\lambda(z), r(z) \in C^1_\alpha(\Gamma), r_1(z) \in C^1(L_2)$, $r_2(y) \in C^2_\alpha(L_4)\,(0 < \alpha < 1)$, we can find two twice continuously differentiable functions $u_0^\pm(z)$ in $\overline{D^\pm}$, for instance, which are the solutions of the Dirichlet problem with the boundary condition on $\Gamma \cup L_2 \cup L_4$ in (1.6) for harmonic equations in $D^\pm$, thus the functions $v(z) = v^\pm(z) = u(z) - u_0^\pm(z)$ in $D$ is the solution of the equation

$$Lv = K_1(y)v_{xx}+K_2(x)v_{yy}+a(x,y)v_x+b(x,y)v_y+c(x,y)v=-\tilde{d}(x,y) \text{ in } D \tag{1.13}$$

satisfying the corresponding boundary conditions

$$\begin{aligned}\mathrm{Re}[\overline{\lambda(z)}W(z)] = R(z) \ \text{ on } \Gamma\cup L_2\cup L_4,\\ v(1)=b_0,\ v(i)=b_3,\ \mathrm{Im}[\overline{\lambda(z)}W(z)]|_{z=z_l}=H(z_l)b_l, l=1,2,\end{aligned} \tag{1.14}$$

where $\tilde{d} = d + Lu_0^{\pm}$, the coefficients of (1.13) satisfy the conditions similar to Condition $C$, $W(z) = U + iV = v_{\tilde{z}}^{+}$ in $D^+$ and $W(z) = U + jV = v_{\tilde{z}}^{-}$ in $\overline{D^-}$, hence later on we only discuss the case of $R(z) = 0$ on $\Gamma \cup L_2 \cup L_4$ and $b_l = 0, l = 0, 1, 2, 3$ in (1.14) and the case of index $K = 0$, which is called Problem $\tilde{P}$, the other case can be similarly discussed. From $v(z) = v^{\pm}(z) = u(z) - u_0^{\pm}(z)$ in $\overline{D^{\pm}}$, we have $u(z) = v^-(z) + u_0^-(z)$ in $\overline{D^-}$, $u(z) = v^+(z) + u_0^+(z)$ in $\overline{D^+}$, $v^+(z) = v^-(z) - u_0^+(z) + u_0^-(z)$ on $L_0 = L_0' \cup L_0''$, and

$$\begin{aligned}&u_y=v_y^{\pm}+u_{0y}^{\pm},\ v_y^{+}=v_y^{-}-u_{0y}^{+}+u_{0y}^{-}=2\hat{R}_1(x), v_y^{-}=2\tilde{R}_1(x) \ \text{ on } L_0'=\{0<x<1, y=0\},\\ &u_x=v_x^{\pm}+u_{0x}^{\pm}, v_x^{+}=v_x^{-}-u_{0x}^{+}+u_{0x}^{-}=2\hat{R}_2(y), v_x^{-}=2\tilde{R}_2(y) \ \text{ on } L_0''=\{x=0, 0<y<1\}.\end{aligned}$$

## 2 Representation of solutions of the oblique derivative problem for mixed equations with nonsmooth degenerate line

In this section, we first write the complex form of equation (1.2). Denote

$$\begin{aligned}&W(z)=U+iV=\frac{1}{2}[H_1(y)u_x-iH_2(x)u_y]=u_{\tilde{z}}=\frac{H_1(y)H_2(x)}{2}[u_X-iu_Y]=H_1(y)H_2(x)u_Z,\\ &H_1(y)H_2(x)W_{\overline{Z}}=\frac{H_1(y)H_2(x)}{2}[W_X+iW_Y]=\frac{1}{2}[H_1(y)W_x+iH_2(x)W_y]=W_{\overline{\tilde{z}}} \ \text{ in } \overline{D^+},\end{aligned} \tag{2.1}$$

we have

$$\begin{aligned}&H_1(y)H_2(x)W_{\overline{Z}}=H_1H_2[W_X+iW_Y]/2=H_1H_2[(U+iV)_X+i(U+iV)_Y]/2\\ &=iH_1H_2[(U+V)-i(U-V)]_{\mu+i\nu}=iH_1H_2\overline{[(U+V)+i(U-V)]_{\mu-i\nu}}=\\ &=\{[iH_2H_{1y}/H_1-a/H_1]H_1u_x-[iH_1H_{2x}/H_2+b/H_2]H_2u_y-cu-d]\}/4\\ &=\{[iH_2H_{1y}/H_1-a/H_1+H_1H_{2x}/H_2-ib/H_2]W+[iH_1H_{2x}/H_2-a/H_1-H_1H_{2x}/H_2\\ &+ib/H_2]\overline{W}-cu-d\}/4=A_1(z)W+A_2(z)\overline{W}+A_3(z)u+A_4(z)=g(Z), \text{ i.e.}\\ &[(U+V)+i(U-V)]_{\mu-i\nu}=\{2[H_2H_{1y}/H_1]U+2[H_1H_{2x}/H_2]V\\ &-i[au_x+bu_y+cu+d]\}/(4H_1H_2)=i\overline{g(Z)} \ \text{ in } D_Z^+=D_Z,\end{aligned} \tag{2.2}$$

in which $D_Z^+, D_\tau^+$ are the image domains of $D^+$ with respect to the mapping $Z = Z(z) = X + iY$, $\tau = \mu + i\nu = \tau(z)$ respectively, and

$$\mu = G_2(x) + G_1(y) = X + Y,\ \nu = G_2(x) - G_1(y) = X - Y \ \text{ in } D^+. \tag{2.3}$$

Similarly introduce the hyperbolic unit $j$ such that $j^2 = 1$, we can obtain

$$W(z)=U+jV=\frac{1}{2}[H_1(y)u_x-jH_2(x)u_y]=\frac{H_1(y)H_2(x)}{2}[u_X-ju_Y]=H_1(y)H_2(x)u_Z,$$

$$H_1(y)H_2(x)W_{\overline{Z}}=\frac{H_1(y)H_2(x)}{2}[W_X+jW_Y]=\frac{1}{2}[H_1(y)W_x+jH_2(x)W_y]=W_{\overline{z}} \text{ in } \overline{D^-},$$

$$-K_1(y)u_{xx}-K_2(x)u_{yy}=H_1(y)[H_1(y)u_x-jH_2(x)u_y]_x+$$

$$+jH_2(x)[H_1(y)u_x-jH_2(x)u_y]_y-jH_2(x)H_{1y}u_x+jH_1(y)H_{2x}u_y=$$

$$=2\{H_1[U+jV]_x+jH_2[U+jV]_y\}-j[H_2H_{1y}/H_1]H_1u_x+$$

$$+j[H_1H_{2x}/H_2]H_2u_y=4H_1(y)H_2(x)W_{\overline{Z}}-j[H_2H_{1y}/H_1]H_1u_x+$$

$$+j[H_1H_{2x}/H_2]H_2u_y=au_x+bu_y+cu+d, \text{ i.e.}$$

$$H_1(y)H_2(x)W_{\overline{Z}}=H_1H_2[W_X+jW_Y]/2=H_1H_2\{(U+V)_\mu e_1+(U-V)_\nu e_2\}$$

$$=\{2j[H_2H_{1y}/H_1]U+2j[H_1H_{2x}/H_2]V+au_x+bu_y+cu+d\}/4=\{[jH_2H_{1y}/H_1$$

$$+a/H_1](W+\overline{W})+[H_1H_{2x}/H_2-jb/H_2](W-\overline{W})+cu+d\}/4=\{[jH_2H_{1y}/H_1$$

$$+a/H_1+H_1H_{2x}/H_2-jb/H_2]W+[jH_2H_{1y}/H_1+a/H_1-H_1H_{2x}/H_2+jb/H_2]\overline{W}$$

$$+cu+d\}/4=\{[a/H_1+H_1H_{2x}/H_2+H_2H_{1y}/H_1-b/H_2](U+V)+[a/H_1-H_1H_{2x}/H_2$$

$$+H_2H_{1y}/H_1+b/H_2](U-V)+cu+d\}e_1/4+\{[a/H_1-H_1H_{2x}/H_2-H_2H_{1y}/H_1-b/H_2]$$

$$\times(U+V)+[a/H_1+H_1H_{2x}/H_2-H_2H_{1y}/H_1+b/H_2](U-V)+cu+d\}e_2/4, \text{ i.e.} \quad (2.4)$$

$$(U+V)_\mu e_1+(U-V)_\nu e_2=[\hat{A}_1(U+V)+\hat{B}_1(U-V)+\hat{C}_1u+\hat{D}_1]e_1+$$

$$+[\hat{A}_2(U+V)+\hat{B}_2(U-V)+\hat{C}_2u+\hat{D}_2]e_2 \text{ in } D_\tau^-,$$

in which $e_1 = (1+j)/2, e_2 = (1-j)/2$, $D_Z^-, D_\tau^-$ are the image sets of $D_1^-$ with respect to the mapping $Z = Z(z), \tau = \mu + j\nu = \tau(z)$ respectively, and

$$\begin{aligned}
\hat{A}_1 &= \frac{1}{4H_1H_2}[\frac{a}{H_1}+\frac{H_1H_{2x}}{H_2}+\frac{H_2H_{1y}}{H_1}-\frac{b}{H_2}], \hat{C}_1=\frac{c}{4H_1H_2},\\
\hat{B}_1 &= \frac{1}{4H_1H_2}[\frac{a}{H_1}-\frac{H_1H_{2x}}{H_2}+\frac{H_2H_{1y}}{H_1}+\frac{b}{H_2}], \hat{C}_2=\frac{c}{4H_1H_2},\\
\hat{A}_2 &= \frac{1}{4H_1H_2}[\frac{a}{H_1}-\frac{H_1H_{2x}}{H_2}-\frac{H_2H_{1y}}{H_1}-\frac{b}{H_2}], \hat{D}_1=\frac{d}{4H_1H_2},\\
\hat{B}_2 &= \frac{1}{4H_1H_2}[\frac{a}{H_1}+\frac{H_1H_{2x}}{H_2}-\frac{H_2H_{1y}}{H_1}+\frac{b}{H_2}], \hat{D}_2=\frac{d}{4H_1H_2} \text{ in } D_1^-.
\end{aligned} \quad (2.5)$$

For the domain $D_2^-$, we can also write the coefficients of equation (2.4) in $D_\tau^-$, where $\tau = \mu + j\nu = G_1(y) + G_2(x) + j[G_1(y) - G_2(x)]$.

It is clear that a special case of (2.2), (2.4) is the complex equation

$$W_{\overline{Z}} = 0 \text{ in } \overline{D_Z}, \quad (2.6)$$

which can be rewritten in the form

$$
\begin{aligned}
&[(U+V)+i(U-V)]_{\mu-i\nu}=0 \ \text{ in } D_\tau^+,\\
&(U+V)_\mu=0,\ (U-V)_\nu=0 \ \text{ in } \overline{D_\tau^-},
\end{aligned}
\tag{2.7}
$$

and the solution $(U+V)+i(U-V)$ is an analytic function in the corresponding domain $D_\tau^+$. The boundary value problem for equations (2.2), (2.4) with the boundary condition (1.14) and the relation: the first formula in (2.9) below will be called Problem $A$.

According to the method in [9], we can verify that there exists a solution of the Riemann-Hilbert problem (Problem $A$) for equation (2.6) in $\overline{D}$ with the boundary conditions

$$
\begin{aligned}
&\mathrm{Re}[\overline{\lambda(z)}(U+iV)]=R(z) \ \text{ on } \Gamma\cup L_2\cup L_4,\\
&\mathrm{Im}[\overline{\lambda(z)}(U+jV)]|_{z=z_l}=b_l, l=1,2,\ u(1)=b_0, u(i)=b_3,
\end{aligned}
\tag{2.8}
$$

in which $\lambda(z)=a(z)+jb(z)$ on $\Gamma\cup L_2\cup L_4$, $R(z)$ is as stated in (1.14).

Now we state and verify the representation of solutions of Problem $P$ for equation (1.2).

**Theorem 2.1** *Under Condition $C$, any solution $u(z)$ of Problem T for equation $(1.2)$ in $\overline{D}$ can be expressed as follows*

$$
\begin{aligned}
&u(z)=u(x)-2\int_0^y\frac{V(z)}{H_2(x)}dy=2\mathrm{Re}\int_1^z\left[\frac{\mathrm{Re}w}{H_1(y)}+\binom{i}{-j}\frac{\mathrm{Im}w}{H_2(x)}\right]dz+b_0 \ \text{ in } \binom{\overline{D^+}}{\overline{D^-}},\\
&w(z)=\Phi(Z)+\Psi(Z)=\hat\Phi(Z)+\hat\Psi(Z),\\
&\Psi(Z)=-2\mathrm{Re}\frac{1}{\pi}\iint_{D_t}\frac{f(t)}{t-Z}d\sigma_t, \hat\Psi(Z)=-2i\,\mathrm{Im}\frac{1}{\pi}\iint_{D_t}\frac{f(t)}{t-Z}d\sigma_t \text{ in } \overline{D_Z}=\overline{D_Z^+},\\
&w(z)=\phi(z)+\psi(z)=\xi(z)e_1+\eta(z)e_2 \ \text{ in } \overline{D^-},\\
&\eta(z)=-\int_0^\nu\frac{g_2(y)}{2H_1(y)}d\nu=\theta(z)+\int_0^y g_2(z)dy=\int_{S_2}g_2(y)dy+\int_0^y g_2(z)dy\\
&=\int_{y_0}^{|y|}g_2(z)dy, z\in s_2, \xi(z)=\zeta(z)+\int_0^y g_1(z)dy,\ z\in s_1,\\
&g_l(z)=\tilde A_l(U+V)+\tilde B_l(U-V)+2\tilde C_lU+\tilde D_lu+\tilde E_l,\ l=1,2,\\
&\xi(z)=\zeta(z)+\int_0^x\hat g_1(z)dx, z\in s_1, \eta(z)=\theta(z)+\int_0^x\hat g_2(z)dx, z\in s_2,\\
&\hat g_l(z)=\hat A_l(U+V)+\hat B_l(U-V)+2\hat C_lU+\hat D_lu+\hat E_l,\ l=1,2,
\end{aligned}
\tag{2.9}
$$

*in which $Z=X+iY=G_2(x)+iG_1(y)$, $f(Z)=g(Z)/H_1H_2$, $U=H_1u_x/2, V=-H_2u_y/2$, $\zeta(z)e_1+\theta(z)e_2$ is a solution of $(2.6)$ in $D_Z^-$, $s_1, s_2$ are two families of characteristics in $D^-$:*

$$
s_1:\ \frac{dx}{dy}=\frac{H_1(y)}{H_2(x)},\ \ s_2:\ \frac{dx}{dy}=-\frac{H_1(y)}{H_2(x)}
\tag{2.10}
$$

*passing through $z=x+jy\in D^-$, $S_1$, $S_2$ are the characteristic curves from the points on $L_1$, $L_2$ to two points on $L_0'$ respectively, $\theta(z)=\int_{S_2}g_2(z)dy$, $\eta(z)=-\int_0^\nu g_2(z)/2H_1(y)d\nu$ is the integral along characteristic curve $s_2$ from a point $z_1=x_1+jy_1$ on $L_2$ to the point $z=x+jy\in\overline{D_Z^-}$,*

*$\theta(x)=-\zeta(x)$ on $L_0'$, and $\zeta(z)=-\theta(G_2(x)-G_1(y))$ on the characteristic curves of $s_1, s_2$ passing through the point $z=x$ respectively, and*

$$w(z)=U(z)+jV(z)=\frac{1}{2}H_1u_x-\frac{j}{2}H_2u_y,$$

$$\xi(z)=\mathrm{Re}\psi(z)+\mathrm{Im}\psi(z),\eta(z)=\mathrm{Re}\psi(z)-\mathrm{Im}\psi(z),$$

$$\tilde{A}_1=\frac{1}{4}[\frac{h_{1y}}{h_1}+\frac{H_1h_{2x}}{H_2h_2}-\frac{2b}{H_2^2}],\ \tilde{B}_1=\frac{1}{4}[\frac{h_{1y}}{h_1}-\frac{H_1h_{2x}}{H_2h_2}+\frac{2b}{H_2^2}],$$

$$\tilde{A}_2=\frac{1}{4}[\frac{h_{1y}}{h_1}+\frac{H_1h_{2x}}{H_2h_2}+\frac{2b}{H_2^2}],\ \tilde{B}_2=\frac{1}{4}[\frac{h_{1y}}{h_1}-\frac{H_1h_{2x}}{H_2h_2}-\frac{2b}{H_2^2}],$$

$$\hat{A}_1=\frac{1}{4}[\frac{H_2h_{1y}}{H_1h_1}+\frac{h_{2x}}{h_2}+\frac{2a}{H_1^2}],\ \hat{B}_1=\frac{1}{4}[\frac{H_2h_{1y}}{H_1h_1}-\frac{h_{2x}}{h_2}+\frac{2a}{H_1^2}],$$

$$\hat{A}_2=\frac{1}{4}[\frac{H_2h_{1y}}{H_1h_1}+\frac{h_{2x}}{h_2}-\frac{2a}{H_1^2}],\hat{B}_2=\frac{1}{4}[\frac{H_2h_{1y}}{H_1h_1}-\frac{h_{2x}}{h_2}-\frac{2a}{H_1^2}]\ \text{ in } D^-,$$

$$\tilde{C}_1=\frac{a}{2H_1H_2}+\frac{m_1}{4y},\tilde{C}_2=-\frac{a}{2H_1H_2}+\frac{m_1}{4y},\tilde{D}_1=-\tilde{D}_2=\frac{c}{2H_2},$$

$$\tilde{E}_1=-\tilde{E}_2=\frac{d}{2H_2}\ \text{ in } D_1^-,\ \hat{C}_1=-\frac{b}{2H_1H_2}+\frac{m_2}{4x},$$

$$\hat{C}_2=\frac{b}{2H_1H_2}+\frac{m_2}{4x},\hat{D}_1=-\hat{D}_2=\frac{c}{2H_1},\hat{E}_1=-\hat{E}_2=\frac{d}{2H_1}\ \text{ in } D_2^-,$$

*in which*

$$H_1(y)=[|y|^{m_1}h_1(y)]^{1/2},\ H_2(x)=[|x|^{m_2}h_1(x)]^{1/2},$$

*herein $h_1(y), h_2(x)$ are continuously differentiable positive functions.*

**Proof** Here and later on we only discuss the integrals in $\overline{D_1^-}$, the case in $\overline{D_2^-}$ can be similarly discussed. From (2.4) it is easy to see that equation (1.2) in $\overline{D_1^-}$ can be reduced to the system of integral equations: (2.9). Moreover we can extend the equation (2.4) onto the the symmetrical domain $\hat{D}_Z$ of $D_Z^-$ with respect to the real axis $\mathrm{Im}Z=0$, namely introduce the function $\hat{W}(Z)$ as follows:

$$\hat{W}(Z)=\begin{cases}W[z(Z)],\\ -\overline{W[z(\overline{Z})]},\end{cases}\ \hat{u}(z)=\begin{cases}u(Z)\ \text{ in } D_Z^-,\\ -u(\overline{Z})\ \text{ in } \hat{D}_Z,\end{cases}$$

and then the equation (2.4) is extended as

$$\hat{W}_{\overline{z}}=\hat{A}_1\hat{W}+\hat{A}_2\overline{\hat{W}}+\hat{A}_3\hat{u}+\hat{A}_4=\hat{g}(Z)\ \text{ in } \overline{D_Z^-}\cup\overline{\hat{D}_Z},$$

where

$$\hat{A}_l(Z)=\begin{cases}A_l(Z),\\ \overline{\tilde{A}_l(\overline{Z})},\end{cases} l=1,2,3,\ \hat{A}_4(Z)=\begin{cases}A_4(Z),\\ -\overline{A_4(\overline{Z})},\end{cases}\ \hat{g}_l(Z)=\begin{cases}g_l(z)\ \text{ in } \overline{D_Z^-},\\ -\overline{g_l(\overline{Z})}\ \text{ in } \overline{\hat{D}_Z},\end{cases}\ l=1,2, \quad (2.11)$$

here $\tilde{A}_1(\overline{Z})=A_2(\overline{Z}), \tilde{A}_2(\overline{Z})=A_1(\overline{Z}), \tilde{A}_3(\overline{Z})=A_3(\overline{Z})$, and we mention that in general $\hat{u}(z)$ on $L_0'$ may not be continuous. It is easy to see that the system of integral equations (2.9) can be written in the form

$$\eta(z)=\theta(z)+\int_0^{\tilde{y}} g_2(z)dy=\int_{y_0}^{\hat{y}} g_2(z)dy,$$
$$\xi(z)=\zeta(z)+\int_0^{y} g_1(z)dy=\int_{y_0}^{\hat{y}} g_1(z)dy,\ \hat{z}=x+j\hat{y}=x+j|y| \text{ in } \tilde{D}_1^-, \tag{2.12}$$

where $x_1+jy_1$ is the intersection point of $L_2$ and the characteristic curve $s_2$ passing through $z=x+jy\in\overline{D_1^-}$, for the extended integral, the function $\zeta(z)$ is determined by $\theta(z)$, which can be appropriately defined in $D_Z^-$, for convenience the above form $g_2(z)$ is written still, and later on the function $\zeta(z)$ will be defined by $\zeta(z)=-\theta(G_2(x)-G_1(y))$, and the numbers $\hat{y}-y_0, \hat{t}-y_0$ will be written by $\tilde{y}, \tilde{t}$ respectively.

## 3 Existence of solutions of the oblique derivative problem for degenerate elliptic equations with nonsmooth degenerate line

For proving the existence of solutions of the oblique derivative problem for mixed equations with nonsmooth degenerate line in $D$, we first give the estimates of the solutions of Problem $\tilde{P}$ for (1.2) in $\overline{D_Z}$. It is clear that Problem $\tilde{P}$ is equivalent to Problem $A$ for the complex equation

$$\begin{aligned} &W_{\overline{Z}}=\frac{1}{H_1H_2}[A_1W+A_2\overline{W}+A_3u+A_4] \text{ in } D_Z,\\ &A_1=\frac{iH_2H_{1y}}{4H_1}+\frac{H_1H_{2x}}{4H_2}-\frac{a}{4H_1}-\frac{ib}{4H_2},\ A_3=\frac{-c}{4},\\ &A_2=\frac{iH_2H_{1y}}{4H_1}-\frac{H_1H_{2x}}{4H_2}-\frac{a}{4H_1}+\frac{ib}{4H_2},\ A_4=\frac{-d}{4}, \end{aligned} \tag{3.1}$$

with the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)}W(z)]=R(z) \text{ on } \Gamma\cup L_2\cup L_4,\ u(1)=b_0,\ u(i)=0 \tag{3.2}$$

and the relation

$$u(z)=u(x)-2\int_0^y\frac{V(z)}{H_2(x)}dy=2\mathrm{Re}\int_1^z\left[\frac{\mathrm{Re}W}{H_1(y)}+i\frac{\mathrm{Im}W}{H_2(x)}\right]dz+b_0. \tag{3.3}$$

As stated in Section 1, we can assume $R(x)=0$ on $\Gamma\cup L_2\cup L_4$ in (3.2), $b_0=u(0)=0,\ u(i)=0$. In the following we first prove that there exists a solution of Problem $A^+$ for (3.1), (3.3) with the boundary condition (3.2) on $\Gamma$ and

$$\mathrm{Re}[-iW(x)]=-H_2(x)\hat{R}_1(x)/2 \text{ on } L_0', \mathrm{Re}[W(iy)]=H_1(y)\hat{R}_2(y)/2 \text{ on } L_0'',$$

and the boundary value problem for (3.1), (3.3) with the boundary condition (3.2) on $L_2\cup L_4$ and

$$\mathrm{Re}[-jW(x)]=H_2(x)\tilde{R}_1(x)/2=R(z) \text{ on } L_0', \mathrm{Re}[W(jy)]=H_1(y)\tilde{R}_2(y)/2=R(z) \text{ on } L_0''$$

will be called Problem $A^-$, where $\hat{R}_1(x),\ \hat{R}_2(y),\ \tilde{R}_1(x),\ \tilde{R}_2(y)$ are as stated in (1.14). From the method in [12], we can prove that Problem $A^+$ for equation (3.1), (3.3) in $D^+$ has a unique solution $W(z)$. Hence in the following we only prove the unique solvability of Problem $A^-$ for (3.1), (3.3) in $D^-$, which is the Darboux type problem (see [2]).

**Theorem 3.1** *If equation* $(1.1)$ *satisfies Condition C, then there exists a solution* $[w(z), u(z)]$ *of Problem* $A^-$ *for* $(3.1)-(3.3)$.

**Proof** We can only discuss in $D_1^-$, because the case in $D_2^-$ can be similarly discussed. By using the method in [10], we may only discuss the problem in $D_* = \overline{D_1^-} \cap \{(0 \le)a_0 = \delta_0 \le x \le a_1 = 1-\delta_0(<1), -\delta \le y \le 0\}$, and $s_1, s_2$ are the characteristics of families in Theorem 2.2 emanating from any two points $(a_0, 0), (a_1, 0)$ $(0 \le a_0 < a_1 < 1)$, where $\delta$, $\delta_0$ are sufficiently small positive numbers. In this case, we can omit the function $K_2(x)$, and may only consider the function $K(y) = K_1(y) = -|y|^m h(y) = -|y|^{m_1} h_1(y)$ and $h_2(x) = 1$, where $m = m_1$, $h(y) = h_1(y)$ is a continuously differentiable positive function in $D_1^-$, $H(y) = \sqrt{|K(y)|}$. It is clear that for two characteristics $s_1$, $s_2$ passing through a point $z = x + jy \in D$ and $x_1, x_2$ are the intersection points with the axis $y = 0$ respectively, for any two points $\tilde{z}_1 = \tilde{x}_1 + j\tilde{y} \in s_1, \tilde{z}_2 = \tilde{x}_2 + j\tilde{y} \in s_2$, we have

$$\begin{aligned}&|\tilde{x}_1 - \tilde{x}_2| \le |x_1 - x_2| = 2|\int_0^y \sqrt{-K(t)}dt| \le \frac{2k_0}{m+2}|y|^{1+m/2}\\ &\le \frac{k_1}{6}|y|^{m/2+1} \le M|y|^{m/2+1}, |y|^{m/2+1} \le \frac{k_0(m+2)}{2}|x_1 - x_2|, -\delta \le y \le 0,\end{aligned} \tag{3.4}$$

where $M$ is a positive constant as stated in (3.6) below. From Condition $C$, we can assume that the coefficients of (2.9) possess continuously differentiable with respect to $x \in L_0'$ and satisfy the conditions

$$\begin{aligned}&|\tilde{A}_l|, |\tilde{A}_{lx}|, |\tilde{B}_l|, |\tilde{B}_{lx}|, |\tilde{D}_l|, |\tilde{D}_{lx}| \le k_0 \le k_1/6,\\ &|\tilde{E}_l|, |\tilde{E}_{lx}| \le k_1/2, 2\sqrt{h}, 1/\sqrt{h}, |h_y/h| \le k_0 \le k_1/6 \text{ in } \bar{D}, l = 1,2,\end{aligned} \tag{3.5}$$

and we shall use the constants

$$\begin{aligned}&M = 4\max[M_1, M_2, M_3, M_4],\ M_1 = \max[8(k_1 d)^2, \frac{M_3}{k_1}],\\ &M_2 = k_1 d + \frac{1}{4}(2\varepsilon_0 + m)|d|^{\beta'-1}, M_3 = 2k_1^2[d + \frac{1}{2H(y_1')}],\\ &M_4 = \frac{1}{H(y_1')}[k_1 d + (\varepsilon_0 + \frac{m}{2})d^{\beta'-1}], M_0 = k_1 d + \frac{2\varepsilon_0 + m}{2\delta} \ge M_2,\\ &\gamma = \max[6k_1\delta + \frac{6\varepsilon(y) + m}{2\beta'} + 2M_4(k_0\delta^{1+m/2})^\beta] < 1,\ 0 \le |y| \le \delta,\end{aligned} \tag{3.6}$$

and $M_l (l = 1, ..., 4)$ are positive constants as stated in (3.9)–(3.14) below, and $d$ is the diameter of $D_1^-$, $1/H(y_1') \le 2k_0[(m+2)a_0/k_0]^{-m/(2+m)}$, $\delta$ is small enough, and $y_1'$ is as stated in (3.10) below. We choose $v_0 = 0, \xi_0 = 0, \eta_0 = 0$ and substitute them into the corresponding positions of $v, \xi, \eta$ in the right-hand sides of (2.9), and obtain

$$\begin{aligned}&v_1(z) = v_1(x) - 2\int_0^y V_0 dy = v_1(x) + \int_0^y (\eta_0 - \xi_0)dy,\\ &\xi_1(z) = \zeta_1(z) + \int_0^y g_{10}(z)dy = \zeta_1(z) + \int_0^y \tilde{E}_1 dy = \int_{y_1}^{\hat{y}} \tilde{E}_1 dy,\\ &\eta_1(z) = \theta_1(z) + \int_0^y g_{20}(z)dy = \theta_1(z) + \int_0^y \hat{E}_2 dy = \int_{y_1}^{\hat{y}} \tilde{E}_2 dy,\\ &g_{l0} = \tilde{A}_l\xi_0 + \tilde{B}_l\eta_0 + \tilde{C}_l(\xi_0 + \eta_0) + \tilde{D}_l v + \tilde{E}_l = \tilde{E}_l, l = 1,2,\end{aligned} \tag{3.7}$$

where $v(x) = u(x) - u_0(x)$ on $L_0'$ as stated before, $z_1 = x_1 + jy_1$ is a point on $L_2$, which is the intersection of $L_2$ and the characteristic curve $s_2$ passing through the point $z = x + jy \in \overline{D_1^-}$. By the successive iteration, we find the sequences of functions $\{v_k\}, \{\xi_k\}, \{\eta_k\}$, which satisfy the relations

$$\begin{aligned} &v_{k+1}(z)=v_{k+1}(x)-2\int_0^y V_k(z)dy=v_{k+1}(x)+\int_0^y(\eta_k-\xi_k)dy,\\ &\xi_{k+1}(z)=\zeta_{k+1}(z)+\int_0^y g_{1k}(z)dy=\int_{y_0}^{\hat{y}} g_{lk}dy,\\ &\eta_{k+1}(z)=\theta_{k+1}(z)+\int_0^y g_{2k}(z)dy=\int_{y_0}^{\hat{y}} g_{2k}(z)dy,\\ &g_{lk}(z)=\tilde{A}_l\xi_k+\tilde{B}_l\eta_k+\tilde{C}_l(\xi_k+\eta_k)+\tilde{D}_l v_k+\tilde{E}_l,\ l=1,2,\ k=0,1,2,..., \end{aligned} \tag{3.8}$$

setting that $\tilde{g}_{lk}(z) = g_{lk}(z) - g_{lk-1}(z)(l = 1, 2)$ and

$$\tilde{y}=\hat{y}-y_1,\tilde{t}=\hat{t}-y_1,\tilde{v}_{k+1}(z)=v_{k+1}(z)-v_k(z),\tilde{\xi}_{k+1}(z)=\xi_{k+1}(z)-\xi_k(z),$$

$$\tilde{\eta}_{k+1}(z)=\eta_{k+1}(z)-\eta_k(z),\tilde{\zeta}_{k+1}(z)=\zeta_{k+1}(z)-\zeta_k(z),\tilde{\theta}_{k+1}(z)=\theta_{k+1}(z)-\theta_k(z),$$

we shall prove that $\{\tilde{v}_k\}, \{\tilde{\xi}_k\}, \{\tilde{\eta}_k\}, \{\tilde{\zeta}_k\}, \{\tilde{\theta}_k\}$ in $D_0$ satisfy the estimates

$$\begin{aligned} &|\tilde{v}_k(z)|,|\tilde{\xi}_k(z)-\tilde{\zeta}_k(z)|,|\tilde{\eta}_k(z)-\tilde{\theta}_k(z)|\le M'\gamma^{k-1}|y|^{1-\beta},\\ &|\tilde{\xi}_k(z)|,|\tilde{\eta}_k(z)|\le M(M_0|\tilde{y}|)^{k-1}/(k-1)!\le M'\gamma^{k-1},\\ &|\tilde{\xi}_k(z_1)-\tilde{\xi}_k(z_2)-\tilde{\zeta}_k(z_1)-\tilde{\zeta}_k(z_2)|,|\tilde{\eta}_k(z_1)-\tilde{\eta}_k(z_2)\\ &-\tilde{\theta}_k(z_1)-\tilde{\theta}_k(z_2)|\le M'\gamma^{k-1}[|x_1-x_2|^{1-\beta}+|x_1-x_2|^{\beta}|t|^{\beta'}],\\ &|\tilde{\xi}_k(z_1)-\tilde{\xi}_k(z_2)|,|\tilde{\eta}_k(z_1)-\tilde{\eta}_k(z_2)|\le M(M_0|\tilde{t}|)^{k-1}[|x_1-x_2|^{1-\beta}\\ &+|x_1-x_2|^{\beta}|\tilde{t}|^{\beta'}]/(k-1)!\le M'\gamma^{k-1}[|x_1-x_2|^{1-\beta}+|x_1-x_2|^{\beta}|t|^{\beta'}],\\ &|\tilde{\xi}_k(z)+\tilde{\eta}_k(z)-\tilde{\zeta}_k(z)-\tilde{\theta}_k(z)|\le M'\gamma^{k-1}|x_1-x_2|^{\beta}|y|^{\beta'},\\ &|\tilde{\xi}_k(z)+\tilde{\eta}_k(z)|\le M(M_0|\tilde{y}|)^{k-1}/(k-1)!\le M'\gamma^{k-1}|x_1-x_2|^{\beta}|y|^{\beta'}, \end{aligned} \tag{3.9}$$

where $z = x + jy, z = x + jt$ is the intersection point of two characteristics of families in (2.10) emanating from two points $z_1, z_2$, $|x_1 - x_2| < 1$, $\beta' = 1 + m/2 - (2 + m)\beta$, $\beta$ is a sufficiently small positive constant, such that $(2 + m)\beta < 1$, moreover $\gamma = \max[6k_1\delta + (6\varepsilon(y) + m)/2\beta' + 2M_4(k_0\delta^{1+m/2})^{\beta}] < 1$, $M_0 = (2k_1d\delta + 2\varepsilon_0 + m)/2\delta$ and $M_4, M'$ are sufficiently large positive constants as stated in (3.6) and (3.11) below.

In fact, from (3.7), it follows that the first formula with $k = 1$ holds, namely

$$\begin{aligned} &|\eta_1(z)-\theta_1(z)|\le|\int_0^y\tilde{E}_2dy|\le|\int_0^y\frac{k_1}{2}dt|\le\frac{k_1|y|}{2},|\theta_1(z)|=|\int_{S_2}\tilde{E}_2dt|\le|\int_{S_2}\frac{k_1}{2}dt|\\ &\le\frac{k_1d}{2},|\eta_1(z)|\le\frac{k_1}{2}|\tilde{y}|\le M,|\xi_1(z)-\zeta_1(z)|\le|\int_0^y\tilde{E}_1dt|\le\frac{k_1|y|}{2}\le M|y|,\\ &|\zeta_1(z)|=|-\theta_1(x-G(y))|\le\frac{k_1d}{2},|\xi_1(z)|\le|\zeta_1(z)|+|\xi_1(z)-\zeta_1(z)|\le\frac{k_1}{2}|\tilde{y}|\le M. \end{aligned}$$

From the first formula in (3.7) and above estimates, we can obtain $|v_1(z)| \le 2d \max_D |\xi_0(z) - \eta_0(z)|/2 \le 2k_1 d^2$. Moreover we can get

$$\begin{aligned}
&|\eta_1(z_1) - \eta_1(z_2) - \theta_1(z_1) + \theta_1(z_2)| \le |\int_0^t [\tilde{E}_2(x_1 + jt) - \tilde{E}_2(x_2 + jt)]dt| \\
&\le |\int_0^t |\tilde{E}_{2x}||x_1 - x_2|dt| \le \frac{k_1}{2} |\int_0^t |x_1 - x_2|dt| \le \frac{k_1}{2}|t||x_1 - x_2| \\
&\le \frac{M|t|}{2}|x_1 - x_2|, |\theta_1(z_1) - \theta_1(z_2)| \le |\int_{S_2} \tilde{E}_2(z)dt - \int_{S_2'} \tilde{E}_2(z)dt| \\
&\le |\int_{S_3} \tilde{E}_{2x}|x_1 - x_2|dt| + |\int_{S_4} \frac{\tilde{E}_2}{2H} d\mu| \le |\int_{y_1'}^0 \frac{k_1}{2}(x_1 - x_2)dt| \\
&+ \frac{k_1}{2H(y_1')}|G(y_2') - G(y_1')| \le \frac{k_1}{2}|y_1'||x_1 - x_2| + \frac{k_1}{4H(y_1')}|x_1 - x_2| \\
&\le \frac{k_1[d + 1/2H(y_1')]}{2}|x_1 - x_2| \le \frac{M_3}{4k_1}|x_1 - x_2| \le \frac{M}{8k_1}|x_1 - x_2| \le M|x_1 - x_2|, \\
&|\xi_1(z_1) - \xi_1(z_2)|, |\eta_1(z_1) - \eta_1(z_2)| \le \frac{M_3}{2k_1}|x_1 - x_2|,
\end{aligned}$$

$$\begin{aligned}
&|\xi_1(z) + \eta_1(z) - \zeta_1(z) - \theta_1(z)| \le |\int_0^y [\tilde{E}_1(z_1) - \tilde{E}_1(z_2)]dt| \\
&\le |\int_0^y \tilde{E}_{1x}[x_1 - x_2]dt| \le \frac{k_1|y|}{2}|x_1 - x_2| \le \frac{M|y|}{8}|x_1 - x_2|^\beta |y|^{\beta'}, \\
&|\zeta_1(z) + \theta_1(z)| = |\theta_1(z) - \theta_1(x - G(y))| \le |\theta_1(x + G(y)) - \theta_1(x - G(y))| \\
&\le \frac{M_3}{2k_1}|x_1 - x_2| \le \frac{M}{8}|x_1 - x_2|^\beta |y|^{\beta'}, |\xi_1(z) + \eta_1(z)| \\
&\le |\zeta_1(z) + \theta_1(z)| + |\xi_1(z) + \eta_1(z) - \zeta_1(z) - \theta_1(z)| \le \frac{M|\tilde{y}|}{4}|x_1 - x_2|^\beta |y|^{\beta'},
\end{aligned} \tag{3.10}$$

where $z = x + jy$ is the same as in (3.12) below, the meaning of the integral $\int_0^\nu [\tilde{E}_2/2H]d\nu$ is as stated in (2.9), $M_3 = 2k_1^2[d + 1/2H(y_1')]$ is a positive constant, $S_2, S_2'$ are two characteristics of family $s_2$ in (2.10) emanating from two points $z_2' = x_2' + jy_2', z_1'' = x_1'' + jy_1''$ on $L_2$ to two points $z_0' = x_0', z_0'' = x_0''(> x_0')$ on $L_0'$, $z_1 = x_1 + jt$, $z_2 = x_2 + jt$ are two points on $S_2, S_2'$ respectively, denote by $z_1' = x_1' + jy_1' = x_1' + jy_1''$ the intersection of $S_2$ and the line $y = y_1' = y_1''$, it is clear that $S_2 = S_3 \cup S_4$, $S_3 = S_2 \cap \{y_1' \le y \le 0\}, S_4 = S_2 \cap \{y < y_1'\}$, and $x_2' + G(y_2') = x_1' + G(y_1') = x_0', x_1'' + G(y_1'') = x_0'', x_2 - x_1 = x_0'' - x_0' = x_1'' - x_1' = 2(x_2' - x_1') = x_2'' - x_1' + G(y_2') - G(y_1'')$, i.e. $G(y_2') - G(y_1'') = x_2' - x_1' = (x_2 - x_1)/2 \le k_1|y_0'|^{(m+2)/2}/(m+2)$, herein $(x_0' + x_0'')/2 + jy_0'$ is the intersection of $S_2$ and $S_1$ emanating from the point $z = x_0''$ on $L_0'$.

In the following we use the inductive method, namely suppose the estimates in (3.9) for $k = n$ are valid, then we can prove that they are true for $k = n + 1$. In the following, we only give the estimates of $\tilde{\xi}_{n+1}(z_1) - \tilde{\xi}_{n+1}(z_2)$, $\tilde{\xi}_{n+1}(z) + \tilde{\eta}_{n+1}(z)$, and $\tilde{\eta}(z) = \eta_{n+1}(z) - \eta_n(z)$, the other estimates can be similarly given. Firstly we estimate the upper bound of $|\tilde{\eta}_{n+1}(z_1) - \tilde{\eta}_{n+1}(z_2)|$. From (3.8), and noting that

$$
\begin{aligned}
&|\tilde{A}_2(z_1)\tilde{\xi}_n(z_1)-\tilde{A}_2(z_2)\tilde{\xi}_n(z_2)|\le|(\tilde{A}_2(z_1)-\tilde{A}_2(z_2))\tilde{\xi}_n(z_1)\\
&+\tilde{A}_2(z_2)(\tilde{\xi}_n(z_1)-\tilde{\xi}_n(z_2))|\le k_1M(M_0|\tilde{t}|)^{n-1}[|x_1-x_2|/6(n-1)!\\
&+|x_1-x_2|^{1-\beta}/6(n-1)!]\le k_1M(M_0|\tilde{t}|)^{n-1}|x_1-x_2|^{1-\beta}/3(n-1)!, t\le-\delta, \text{ or}\\
&\le k_1M'\gamma^{n-1}[|x_1-x_2|/6+|x_1-x_2|^{\beta}|t|^{\beta'}/6], -\delta\le t\le 0,\\
&|\tilde{C}_2(z_1)(\tilde{\xi}_n(z_1)+\tilde{\eta}_n(z_1))-\tilde{C}_2(z_2)(\tilde{\xi}_n(z_2)+\tilde{\eta}_n(z_2))|+\\
&\le|\frac{1}{2H}|(a(z_1)-a(z_2))(\tilde{\xi}_n(z_1)+\tilde{\eta}_n(z_1))+\\
&+a(z_2)(\tilde{\xi}_n(z_1)+\tilde{\eta}_n(z_1)-\tilde{\xi}_n(z_2)-\tilde{\eta}_n(z_2))|\\
&+|\frac{m}{4t}(\tilde{\xi}_n(z_1)+\tilde{\eta}_n(z_1)-\tilde{\xi}_n(z_2)-\tilde{\eta}_n(z_2))|\\
&\le\frac{M(M_0|\tilde{t}|)^{n-1}}{(n-1)!}\frac{4\varepsilon(t)+m}{2|t|}|x_1-x_2|^{1-\beta},\ t\le-\delta, \text{ or}\\
&\le M'\gamma^{n-1}|x_1-x_2|^{\beta}\frac{4\varepsilon(t)+m}{2|t|}|t|^{\beta'},\ -\delta\le t\le 0,
\end{aligned}
$$

we can get

$$
\begin{aligned}
&|\tilde{\eta}_{n+1}(z_1)-\tilde{\eta}_{n+1}(z_2)-\tilde{\theta}_{n+1}(z_1)+\tilde{\theta}_{n+1}(z_2)|\\
&\le|\int_0^t[\tilde{g}_{2n}(z_1)-\tilde{g}_{2n}(z_2)]dt|\le|\int_0^t M'\gamma^{n-1}[2k_1|x_1-x_2|\\
&+|x_1-x_2|^{\beta}(2k_1+\frac{4\varepsilon(t)+m}{2|t|})|t|^{\beta'}]dt|\le M'\gamma^{n-1}[2k_1|x_1-x_2|t|\\
&+|x_1-x_2|^{\beta}(2k_1|t|+\frac{4\varepsilon(t)+m}{2\beta'})|t|^{\beta'}]\le M'\gamma^{n}|x_1-x_2|^{\beta}|t|^{\beta'},\ -\delta\le t\le 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\tilde{\eta}_{n+1}(z_1)-\tilde{\eta}_{n+1}(z_2)-\int_{-\delta}^t[\tilde{g}_{2n}(z_1)-\tilde{g}_{2n}(z_2)]dt|\\
&\le|\int_{y_1}^{-\delta}[\tilde{g}_{2n}(z_1)-\tilde{g}_{2n}(z_2)]dt|\le|\int_{y_1}^{-\delta}\frac{M(M_0|t|)^{n-1}}{(n-1)!}[k_1|x_1-x_2|^{1-\beta}\\
&+\frac{2\varepsilon(t)+m}{2|t|}|x_1-x_2|^{1-\beta}]dt|+|\int_{S_4}\frac{MM_4(M_0|t|)^{n-1}}{(n-1)!}d\mu|\\
&\le\frac{M(M_0|\tilde{t}|)^{n-1}}{(n-1)!}[\frac{|\tilde{t}|}{n}(k_1|x_1-x_2|^{1-\beta}+\frac{2\varepsilon_0+m}{2\delta}|x_1-x_2|^{1-\beta})\\
&+2M_4|G(y_2')-G(y_1')|]\le M\frac{(M_0|\tilde{t}|)^{n-1}}{(n-1)!}[\frac{M_0|t|}{n}|x_1-x_2|^{1-\beta}+M_4|x_1-x_2|]\\
&\le M'\gamma^{n-1}[\gamma|x_1-x_2|^{1-\beta}+M_4|x_1-x_2|]\le M'\gamma^{n}|x_1-x_2|^{\beta}|t|^{\beta'}, t\le-\delta,
\end{aligned}
\tag{3.11}
$$

where $\tilde{t}=|t|-y_1$, $|x_1-x_2|<1$, $M_0, M_4$ are as stated in (3.6), and $M_4=[k_1d+(\varepsilon_0+m/2)d^{\beta'-1}]/H(y_1')\le 2k_0[k_1d+(\varepsilon_0+m/2)d^{\beta'-1}][(m+2)a_0/k_0]^{-m/(2+m)}$, $\beta$ is a sufficiently small positive constant, such that $2M_4|x_1-x_2|\le k_1|x_1-x_2|^{1-\beta}$, similarly to the integral $\int_0^y[\tilde{g}_{2n}(z_1)-\tilde{g}_{2n}(z_2)]dy$, the estimate of $\int_{-\delta}^{|y|}[\tilde{g}_{2n}(z_1)-\tilde{g}_{2n}(z_2)]dy$ is given, and $2M_4|G(y_2')-G(y_1')|=M_4|x_1-x_2|$ and $\int_{-\delta}^0[\tilde{g}_{2n}(z_1)-\tilde{g}_{2n}(z_2)]dt$ are estimated together with $\int_0^y[\tilde{g}_{2n}(z_1)-\tilde{g}_{2n}(z_2)]dt$. It is clear that there exists a sufficiently large positive integer $N$ such that $2dM_0/(N+1)\le\gamma$, hence we can discuss the problem only for $n>N$. But the constant $M$ can be replaced by a sufficiently large positive constant

$M' = M\max_{0\le n\le N}(M_0|2d|)^n/(n!\gamma^n)$, then we have $M(M_0|\tilde{t}|)^n/n! \le M'\gamma^n \le M'$, $n=1,2,\ldots$. Secondly we consider

$$I=I_1+I_2,\ I_1=\tilde{\xi}_{n+1}(z)+\tilde{\eta}_{n+1}(z)-\tilde{\zeta}_{n+1}(z)-\tilde{\theta}_{n+1}(z), I_2=\tilde{\zeta}_{n+1}(z)+\tilde{\theta}_{n+1}(z),$$

noting that

$$\begin{aligned}
&|\tilde{A}_1(z_1)\tilde{\xi}_n(z_1)+\tilde{A}_2(z_2)\tilde{\xi}_n(z_2)+\tilde{B}_1(z_1)\tilde{\eta}_n(z_1)+\tilde{B}_2(z_2)\tilde{\eta}_n(z_2)|\\
&\le|[\tilde{A}_1(z_1)-\tilde{A}_1(z_2)]\tilde{\xi}_n(z_1)+[\tilde{A}_1(z_2)+\tilde{A}_2(z_2)]\tilde{\xi}_n(z_1)+\\
&+\tilde{A}_2(z_2)[\tilde{\xi}_n(z_2)-\tilde{\xi}_n(z_1)]+[\tilde{B}_1(z_1)-\tilde{B}_1(z_2)]\tilde{\eta}_n(z_1)\\
&+[\tilde{B}_1(z_2)+\tilde{B}_2(z_2)]\tilde{\eta}_n(z_1)|+\tilde{B}_2(z_2)[\tilde{\eta}_n(z_2)-\tilde{\eta}_n(z_1)]|\\
&\le 2k_1M(M_0|t|)^{n-1}(|x_1-x_2|^{1-\beta}+|x_1-x_2|^{\beta}|t|^{\beta'})/3(n-1)!=I_3,\\
&|\tilde{C}_1(z_1)(\tilde{\xi}_n(z_1)+\tilde{\eta}_n(z_1))+\tilde{C}_2(z_2)(\tilde{\xi}_n(z_2)+\tilde{\eta}_n(z_2))|\\
&=|\tilde{C}_2(z_2)[\tilde{\xi}_n(z_2)+\tilde{\eta}_n(z_2)+\tilde{\xi}_n(z_1)+\tilde{\eta}_n(z_1)]+[\tilde{C}_1(z_1)-\tilde{C}_2(z_2)]\\
&\times[\tilde{\xi}_n(z_1)+\tilde{\eta}_n(z_1)]|\le M\frac{(M_0|t|)^{n-1}}{(n-1)!}|x_1-x_2|^{\beta}(\frac{2\varepsilon(t)}{|t|}+\frac{m}{2|t|})|t|^{\beta'}=I_4,
\end{aligned}$$

the inequality

$$\begin{aligned}
&|I_1|=|\int_0^y[\tilde{g}_{1n}(z_1)+\tilde{g}_{2n}(z_2)]dt|\le|\int_0^y[\tilde{g}_{1n}(z_1)+\tilde{g}_{2n}(z_2)dt|\\
&\le|\int_0^y M'\gamma^{n-1}[3k_1|x_1-x_2|^{1-\beta}+|x_1-x_2|^{\beta}(2k_1+\frac{6\varepsilon(t)+m}{2|t|}|t|^{\beta'}]dt|\\
&\le M'\gamma^{n-1}[3k_1|x_1-x_2|^{1-\beta}|y|+|x_1-x_2|^{\beta}(2k_1|y|+\frac{6\varepsilon(y)+m}{2\beta'})|y|^{\beta'}]\\
&\le M'\gamma^n|x_1-x_2|^{\beta}|y|^{\beta'},\ -\delta\le y\le 0,
\end{aligned}\tag{3.12}$$

and

$$|I|=|\int_{y_1}^{\hat{y}}[\hat{g}_{1n}(z)+\hat{g}_{2n}(z)]dy|\le M'\gamma^n|x_1-x_2|^{\beta}|y|^{\beta'}$$

can be derived, where $z=x+jy$, $2M_4|x_1-x_2|\le k_1|x_1-x_2|^{1-\beta}$ in (3.11), and the estimate of $|I_2|$ is used by the same way in (3.11). Finally we estimate

$$\begin{aligned}
II=II_1+II_2, II_1&=\tilde{\eta}_{n+1}(z)-\tilde{\theta}_{n+1}(z)=\int_0^y\tilde{g}_{2n}(z)dy\\
&=\int_0^y[\tilde{A}_2\tilde{\xi}_n+\tilde{B}_2\tilde{\eta}_n+\tilde{C}_2(\tilde{\xi}_n+\tilde{\eta}_n)+\tilde{D}_2\tilde{v}_n]dy,\ z\in s_2,
\end{aligned}$$

and can get

$$\begin{aligned}
&|II_1|=|\int_0^y\tilde{g}_{2n}(z)dy|\le|\int_0^y M\frac{(M_0|y|)^{n-1}}{(n-1)!}[k_1|x_1-x_2|^{1-\beta}+\frac{2\varepsilon(t)+m}{4|t|}|t|^{\beta'}]dt|\\
&\le M\frac{(M_0|y|)^n}{n!}|y|^{1-\beta}, |II_2|=|\theta_{n+1}(x-G(y))|\le M\frac{M_2M_0^{n-1}d^n}{n!},\\
&|II|=|II_1|+|II_2|\le M\frac{(M_0|\tilde{y}|)^n}{n!},
\end{aligned}\tag{3.13}$$

where $M_2 = k_1 d + (\varepsilon_0/2+m/4)d^{\beta'-1} \le M_0$. Here we mention that the functions $\tilde{\eta}_{n-1}(z)$, $\tilde{\xi}_{n-1}(z) + \tilde{\eta}_{n-1}(z), \tilde{\xi}_{n-1}(z)$ may successively choose, when we estimate the upper bound of above functions $\tilde{\eta}_n(z)$, $\tilde{\xi}_n(z)$, $\tilde{\xi}_n(z) + \tilde{\eta}_n(z)$. As for the estimate in the neighborhood $D' = \overline{D_1^-} \cap \{a_1 = 1 - \delta_0 \le x \le 1\}(\subset \{-\delta \le y \le 0\})$ of $z = 1$, which can be obtained by the simpler successive iteration, because it is belonging the case: $-\delta \le y \le 0$, when $\delta_0$ is small enough.

On the basis of the estimate (3.9), the convergence of sequences $\{M(M_0|\tilde{y}|)^{k-1}/(k-1)!\}$, $\{M'\gamma^{k-1}|y|^{\beta'}\}$, and the comparison test, we can derive that $\{v_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ in $\overline{D_1^-} \cap \{-\delta \le y \le 0\}$ uniformly converge to $v_*$, $\xi_*$, $\eta_*$ satisfying the system of integral equations: (2.9), and by the result in [10], we can derive that the function $[W_*(z), v_*(z)] = [(\xi_* + \eta_* + j\xi_* - j\eta_*)/2, v_*(z)]$ is a solution of Problem $A^-$ for equation (3.1) in $\overline{D_1^-} \cap \{y < -\delta\}$. Moreover the function $u(z) = v_*(z) + u_0(z)$ is a solution of Problem $P$ for (1.2) in $D^-$. The proof is finished.

From the above discussion and by using the method in [11], [12], we can obtain the following theorem.

**Theorem 3.2** *Let equation* (1.2) *satisfy Condition C. Then the above oblique derivative problem* (*Problem* $P$) *for* (1.2) *has a unique solution.*

In [6], the author mainly investigated the unique solvability of Tricomi problem for equation (1.2) with $K_2(x) = 1, a = b = c = d = 0$ in $D$. In [7], the author discussed the uniqueness and existence of solutions of homogeneous Tricomi problem for second order equation of mixed type with parabolic degenerate line by using the method of energy integral, but he assumes that the coefficients of (1.2) satisfy some stronger conditions, for instance $K_2(x) = 1$, $b \ge 0$, $c \le 0$, $K_y - a\max_{L_2}\sqrt{-K} > 0$, $c_y - \max_{L_2}\sqrt{-K}c_x \le 0$ in $D$, and the inner angles of elliptic domain $D^+$ at $z = 0, 2$ are less than $\pi/2$. In [5], the authors discuss some Tricomi problem for equation (1.2) with the condition $K_1(y) = \mathrm{sgn} y|y|^m$, $K_2(x) = x^n$, $\eta = x^p|y|^q\eta^*(x,y)$, $\eta = a,b,c,d\,(\eta^* \in C^1(\overline{D}), m,n,p,q$ are positive numbers) by the method of integral equations. In the present paper, we consider the equation (1.2) with Condition $C$, which is weaker then the conditions as stated in [2]-[7].

Finally we mention that the coefficients $K_1(y), K_2(x)$ in equation (1.2) can be replaced by functions $K_l(x,y)\,(l = 1,2)$ with some conditions, for instance $K_1(x,y) = \mathrm{sgn}y|y|^{m_1}h_1(x,y)$, $K_2(x,y) = \mathrm{sgn}x|x|^{m_2}h_2(x,y)$, $h_l(x,y)\,(l = 1,2)$ are continuously differentiable positive functions. Besides if the boundary condition on $L_2 \cup L_4$ in (1.6) is replaced by the boundary condition $L_1 \cup L_3$ in (1.6), then we can also discuss by the similar method. The equation (1.2) in this paper includes the mixed (bi-hyperbolic elliptic and bi-parabolic) equation

$$K_1(y)[M_2(x)u_x]_x + M_1(x)[K_2(y)u_y]_y + ru = f \text{ in } D,$$

in [4] as a special case. In fact the above equation can be rewritten as

$$K_1(y)M_2(x)u_{xx}+M_1(x)K_2(y)u_{yy}+K_1(y)M_{2x}u_x+M_1(x)K_{2y}u_y+ru=f \text{ in } D,$$

the above equation is divided by $K_2M_2(K_2 > 0, M_2 > 0)$ in $D$, we obtain

$$Ku_{xx} + Mu_{yy} + K(\ln M_2)'u_x/M_2 + M(\ln K_2)'u_y/K_2 + ru/M_2K_2 = f/M_2K_2,$$

where $K = K_1/K_2, M = M_1/M_2$.

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