Ulam Stability Problem

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- In 1940, S. M. Ulam proposed, before the Mathematics Club of the University of Wisconsin, the following problem on homomorphisms.

Ulam Problem: given a group G_1 and a metric group G_2 with metric $d(.,.)$ as well as $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G_1$?

In 1968, Ulam ("a collection of mathematical problems", Intersci. Publ., Inc., New York, 1968, p. 63) posed the following more general problem.

General Ulam Problem: When is it true that by changing slightly the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?

D. H. Hyers (Proc. Nat. Acad. Sci., 27 (1941), no. 4, 222-224) considered the case of approximately additive mappings $f : X \to Y$ satisfying the functional inequality $\| f(x+y) - f(x) - f(y) \| < \varepsilon$ for all x, y $\in X$, where X and Y are Banach spaces. Then he showed that $A(x) = \lim\limits_{n \to \infty} 2^{-n} f(2^n x)$ exists for all $x \in X$ and that $A : X \to Y$ is the unique additive mapping satisfying $\| f(x) - A(x) \| \le \varepsilon$ for all $x \in X$ with $\varepsilon > 0$.

According to P. M. Gruber (Trans. Amer. Math. Soc. 245 (1978), 263-277) this Ulam problem is of particular interest in probability theory and in the case of functional equations of different types. There is now an extensive research work on the stability of the Ulam problem for functional equations with applications in probability theory, financial and actuarial mathematics.