The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 1, Issue 1, Article 10, pp. 1-21, 2004

ASYMPTOTIC BEHAVIOR OF MIXED TYPE FUNCTIONAL EQUATIONS

J. M. RASSIAS

Received 13 May 2004; accepted 7 June 2004; published 1 August 2004;

PEDAGOGICAL DEPARTMENT, E.E., NATIONAL AND CAPODISTRIAN UNIVERSITY OF ATHENS,
SECTION OF MATHEMATICS AND INFORMATICS, 4, AGAMEMNONOS STR., AGHIA PARASKEVI,
ATHENS 15342,GREECE
jrassias@primedu.uoa.gr
URL:http://www.primedu.uoa.gr/~jrassias/

ABSTRACT. In 1983 Skof [24] was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 Jung [14] investigated the Hyers-Ulam stability of additive and quadratic mappings on restricted domains. In this paper we improve the bounds and thus the results obtained by Jung [14], in 1998 and by the author [21], in 2002. Besides we establish new theorems about the Ulam stability of mixed type functional equations on restricted domains. Finally, we apply our recent results to the asymptotic behavior of functional equations of different types.

Keywords and phrases: Ulam stability; Asymptoticity; Mixed type mapping; Restricted domain.

2000 Mathematics Subject Classification: Primary 39B. Secondary 26D.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In 1940 and in 1964 Ulam [25] proposed the *general Ulam stability problem*:

> "When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In 1941 Hyers [13] solved this problem for linear mappings. In 1951 Bourgin [3] was the second author to treat the Ulam problem for additive mappings. In 1978, according to Gruber [12], *this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types.* In 1999 Gavruta [11] answered a question of the author [18] concerning the stability of the Cauchy equation. For some other interesting stability results in connection to the Ulam problem, see [1], [2, 9], [4], [5], [6], [7, 8], [10], [15], [16-21], [22] and [23]. In 1983 Skof [24] was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 Jung [14] and in 2002-2003 the author [21, 22] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. In this paper we improve the bounds and thus the stability results obtained by Jung, in 1998 and by the author, in 2002. Besides we establish new theorems about the Ulam stability for more general equations of two types on a restricted domain. Finally we apply our recent results to the asymptotic behavior of functional equations of different types.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations.

Definition 1.1. A mapping $f : X \to Y$ is called *additive* (respectively: *quadratic*) if f satisfies the equation

$$f(x_1 + x_2) = f(x_1) + f(x_2) \tag{1.1}$$

(respectively: $f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2)$)
for all $x_1, x_2 \in X$.

We state and prove the new Theorem 1.1 below.

Theorem 1.1. *Let $\delta \geq 0$ be fixed. If a mapping $f : X \to Y$ satisfies the quadratic inequality*

$$\left\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2) \right\| \leq \delta \tag{1.2}$$

for all $x_1, x_2 \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that $||f(0)|| \leq \delta / 2$ and

$$\left\| f(x) - Q(x) \right\| \leq \frac{\delta + \| f(0) \|}{3} (\leq (\delta + \frac{\delta}{2})/3 = \delta/2) \tag{1.3}$$

for all $x \in X$.

Proof. Replacing $x_1 = x_2 = 0$ in the inequality (1.2), one obtains that

$$||f(0)|| \leq \frac{\delta}{2}$$

holds. Similarly, substituting $x_1 = x_2 = x$ in (1.2) and then applying the triangle inequality, we get the inequality

$$||f(x)-2^{-2}f(2x)|| \le \frac{\delta+\|f(0)\|}{3}(1-2^{-2})$$

for a fixed $\delta \ge 0$ and all $x \in X$.

According to our works [19, 20] on quadratic mappings, one proves that

$$||f(x)-2^{-2n}f(2^n x)|| \le ||f(x)-2^{-2}f(2x)|| + 2^{-2}||f(2x)-2^{-2}f(2^2 x)|| + \ldots$$

$$+ 2^{-2(n-1)}||f(2^{n-1}x)-2^{-2}f(2^n x)||$$

$$\le \frac{\delta+\|f(0)\|}{3}(1-2^{-2n})$$

holds for all $n \in N$, and all $x \in X$, which yields that there is a unique quadratic mapping $Q: X \to Y$, such that

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x),$$

and

$$\|f(x)-Q(x)\| \le \frac{\delta+\|f(0)\|}{3},$$

completing the proof of the Theorem 1.1. ■

Definition 1.2. A mapping $f : X \to Y$ is called *approximately odd* (respectively: *even*) if f satisfies

$$\|f(x)+f(-x)\| \le \theta \tag{1.4}$$

(respectively : $\|f(x)-f(-x)\| \le \theta$) for some fixed $\theta \ge 0$ and for all $x \in X$.

Definition 1.3. A mapping $M : X \to Y$ is called *additive* (respectively*: quadratic*) in X if M satisfies the functional equation of two types

$$M\left(\sum_{i=1}^{3} x_i\right) + \sum_{i=1}^{3} M(x_i) = \sum_{1\le i\langle j\le 3} M(x_i + x_j) \tag{1.5}$$

for all $x_i \in X$ (i = 1, 2, 3). We note that all the real mappings $M : R \to R$ of the two types: $M(x) = ax$ or $M(x) = \beta x^2$ satisfy (1.5) for all $x \in R$ and all arbitrary but fixed $a, \beta \in R$.

We note that the mapping $M : X \to Y$ may be called *mixed type* as it is either additive or quadratic. The same terminology occurs to the mappings M satisfying the following equation (4.11).

2. Stability of the Quadratic Equation (1.1) on a Restricted Domain

We establish the new Theorem 2.1 on restricted domains.

Theorem 2.1. *Let* $d > 0$ *and* $\delta \ge 0$ *be fixed. If a mapping* $f : X \to Y$ *satisfies the quadratic inequality* (1.2) *for all* $x_1, x_2 \in X$, *with* $\|x_1\| + \|x_2\| \ge d$, *then there exists a unique quadratic mapping* $Q : X \to Y$ *such that* $||f(0)|| \le \delta/2$ *and*

$$\|f(x)-Q(x)\| \le \frac{9\delta+4\|f(0)\|}{6} \left(\le \frac{11}{6}\delta\right) \tag{2.1}$$

for all $x \in X$.

Proof. Assume $\|x_1\|+\|x_2\|< d$. If $x_1 = x_2 = 0$, then we choose a $t \in X$ with $\| t \| = d$. Otherwise, let

$$t=\left(1+\frac{d}{\|x_1\|}\right)x_1 \;,\; if \quad \|x_1\| \geq \|x_2\|;\; t=\left(1+\frac{d}{\|x_2\|}\right)x_2 \;,\; if \quad \|x_1\| \leq \|x_2\|.$$

We note that: $\| t \| = \|x_1\| + d > d$, if $\|x_1\| \geq \|x_2\|$; $\| t \| = \|x_2\| + d > d$, if $\|x_1\| \leq \|x_2\|$.

Clearly, we see

$$\|x_1 - t\| + \|x_2 + t\| \geq 2\| t \| - (\|x_1\| + \|x_2\|) \geq d \,, \|x_1 - x_2\| + \| 2t \| \geq \|x_1 - x_2\| + 2d \geq d \,,$$

$$\|x_1 + t\| + \|-x_2 + t\| \geq d\,,\; \|x_1\| + \| t \| \geq d \;,\; \| t \| + \|x_2\| \geq d\,,\; \| t \| + \| t \| \geq d\,. \tag{2.2}$$

These inequalities (2.2) come from the corresponding substitutions attached between the right-hand sided parentheses of the following functional identity.

Besides from (1.2) with $x_1 = x_2 = 0$ we get that $\|f(0)\| \leq \delta/2$. Therefore from (1.2), (2.2), and the new *functional identity*

$$2[f(x_1+x_2)+f(x_1-x_2)-2f(x_1)-2f(x_2)-f(0)]$$
$$=[f(x_1+x_2)+f(x_1-x_2-2t)-2f(x_1-t)-2f(x_2+t)] \text{ (with } x_1-t \text{ on } x_1 \text{ and } x_2+t \text{ on } x_2)$$
$$-[f(x_1-x_2-2t)+f(x_1-x_2+2t)-2f(x_1-x_2)-2f(2t)] \text{ (with } x_1-x_2 \text{ on } x_1 \text{ and } 2t \text{ on } x_2)$$
$$+[f(x_1-x_2+2t)+f(x_1+x_2)-2f(x_1+t)-2f(-x_2+t)] \text{ (with } x_1+t \text{ on } x_1 \text{ and } -x_2+t \text{ on } x_2)$$
$$+2[f(x_1+t)+f(x_1-t)-2f(x_1)-2f(t)] \text{ (with } x_1 \text{ on } x_1 \text{ and } t \text{ on } x_2)$$
$$+2[f(t+x_2)+f(t-x_2)-2f(t)-2f(x_2)] \text{ (with } t \text{ on } x_1 \text{ and } x_2 \text{ on } x_2)$$
$$-2[f(2t)+f(0)-2f(t)-2f(t)] \text{ (with } t \text{ on } x_1 \text{ and } t \text{ on } x_2),$$

we get

$$2\|f(x_1+x_2)+f(x_1-x_2)-2f(x_1)-2f(x_2)-f(0)\| \leq \delta+\delta+\delta+2\delta+2\delta+2\delta = 9\delta,$$

or

$$\|f(x_1+x_2)+f(x_1-x_2)-2f(x_1)-2f(x_2)\| \leq \frac{9}{2}\delta + \|f(0)\| \; (\leq 5\delta). \tag{2.3}$$

Applying now the Theorem 1.1 and the above inequality (2.3), one gets that

$$\|f(x)-Q(x)\| \leq [\,(\frac{9}{2}\delta + \| f(0) \|) + \| f(0) \|]/3$$
$$= \frac{3}{2}\delta + \frac{2}{3}\| f(0) \| = \frac{9\delta + 4\| f(0) \|}{6}$$
$$(\leq \frac{9\delta + 2\delta}{6} = \frac{11}{6}\delta).$$

Therefore there exists a unique quadratic mapping $Q : X \to Y$ that satisfies the quadratic equation (1.1) and the inequality (2.1), such that $Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$, completing the proof of the Theorem 2.1. ∎

Obviously our inequalities (2.1) and (2.3) are sharper than the corresponding inequalities of Jung [14], where the right-hand sides were equal to

$$\frac{7}{2}\delta \;\; \geq \frac{9\delta + 4\| f(0) \|}{6} (\leq \frac{11}{6}\delta) \quad \text{and} \quad 7\delta (\geq \frac{9}{2}\delta + \| f(0) \|)\,(\leq 5\delta),$$

respectively.

We note that if we define $S_2 = \{(x_1, x_2) \in X^2 : \|x_i\| \langle d, i = 1, 2\}$ for some $d > 0$, then

$$\{(x_1, x_2) \in X^2 : \|x_1\| + \|x_2\| \geq 2d\} \subset X^2 \setminus S_2.$$

Corollary 2.1. *If we assume that a mapping $f : X \to Y$ satisfies the quadratic inequality* (1.2) *for some fixed $\delta \geq 0$ and for all $(x_1, x_2) \in X^2 \setminus S_2$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (2.1) *for all $x \in X$.*

Corollary 2.2. *A mapping $f : X \to Y$ is quadratic if and only if the asymptotic condition*

$$\|f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2)\| \to 0, \text{ as } \|x_1\| + \|x_2\| \to \infty$$

holds.

Proof. Following the corresponding techniques of our proof [21], in 2002, one gets from Theorem 2.1 and the above asymptotic condition that f is quadratic. The reverse assertion is obvious. █

However, in 1983 Skof [24] proved an asymptotic property of the additive mappings.

3. STABILITY OF THE EQUATION (1.5) OF TWO TYPES

In 1998 Jung [14] applied the induction principle and proved the following Lemma 3.1.

Lemma 3.1. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3) + \sum_{i=1}^{3} f(x_i) \right\| \leq \delta$$

for some fixed $\delta \geq 0$ and for all $x_i \in X$ ($i = 1, 2, 3$). It then holds that

$$\left\| f(x) - \frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x) \right\| \leq 3\delta \sum_{i=1}^{n} 2^{-i},$$

for all $x \in X$ and $n \in N = \{1, 2, \ldots\}$.

In 2002 the author [21] proved Lemma 3.1 without the induction principle.

In this paper we establish the following sharper Lemma 3.2, because of $\|f(0)\| \leq \delta$ via (3.5).

Lemma 3.2. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3) + \sum_{i=1}^{3} f(x_i) \right\| \leq \delta \tag{3.1}$$

for some fixed $\delta \geq 0$ and for all $x_i \in X$ ($i = 1, 2, 3$). It then holds that

$$\left\| f(x) - \frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x) \right\| \leq (\delta + 2\|f(0)\|) \sum_{i=1}^{n} 2^{-i} \left(\leq 3\delta \sum_{i=1}^{n} 2^{-i}\right), \tag{3.2}$$

for a fixed $\delta \geq 0$ *and all* $x \in X$ *and* $n \in N = \{1, 2, \ldots\}$,*such that* $||f(0)|| \leq \delta$.

In this paper, *without the induction principle* and following the proof of our Lemma 1 [21], we prove the above-mentioned Lemma 3.2.

Proof. Substituting $x_i = 0$ for all i=1,2,3, in the inequality (3.1), one obtains $||f(0)|| \leq \delta$.

Let us denote

$$a_i = \frac{2^i + 1}{2^{2i+1}},\ A_i(x) = 3f\left(2^{i-1}x\right) + f\left(-2^{i-1}x\right) - f\left(2^i x\right),$$

$$b_i = -\frac{2^i - 1}{2^{2i+1}},\ B_i(x) = 3f\left(-2^{i-1}x\right) + f\left(2^{i-1}x\right) - f\left(-2^i x\right),$$

$$T_i(x) = a_i f\left(2^i x\right) + b_i f\left(-2^i x\right),\ S_n(x) = T_0(x) - T_n(x),$$

such that $T_0(x) = f(x)$, for all $x \in X$, $i \in N_n = \{1, 2, \ldots, n\}$, and $n \in N$.

We note that

$$a_{i-1} = 3a_i + b_i,\ \ b_{i-1} = a_i + 3b_i,$$

hold for any $i \in N_n = \{1, 2, \ldots, n\}$, $n \in N$.

From these identities we get that

$$T_{i-1}(x) - T_i(x) = a_{i-1} f\left(2^{i-1}x\right) + b_{i-1} f\left(-2^{i-1}x\right) - T_i(x)$$

$$= (3a_i + b_i) f\left(2^{i-1}x\right) + (a_i + 3b_i) f\left(-2^{i-1}x\right) - a_i f\left(2^i x\right) - b_i f\left(-2^i x\right)$$

$$= a_i\left[3f\left(2^{i-1}x\right) + f\left(-2^{i-1}x\right) - f\left(2^i x\right)\right] + b_i\left[3f\left(-2^{i-1}x\right) + f\left(2^{i-1}x\right) - f\left(-2^i x\right)\right],$$

or the formula

$$T_{i-1}(x) - T_i(x) = a_i A_i(x) + b_i B_i(x)\ , \tag{3.3}$$

holds for any $i \in N_n = \{1, 2, \ldots, n\}$, $n \in N$.

We note that

$$S_n(x) = T_0(x) - T_n(x) = \sum_{i=1}^{n}\left[T_{i-1}(x) - T_i(x)\right].$$

Therefore from this formula and (3.3) one obtains the new formula

$$S_n(x) = \sum_{i=1}^{n}\left[a_i A_i(x) + b_i B_i(x)\right]. \tag{3.4}$$

Replacing $x_i = 0$ ($i = 1, 2, 3$) in (3.1) one gets

$$\|f(0)\| \leq \delta. \tag{3.5}$$

Setting $x_1 = x$, $x_2 = x$, $x_3 = -x$ in (3.1) we find from (3.5) that

$$\|3f(x) + f(-x) - f(2x) - 2f(0)\| \leq \delta$$

or

$$\|3f(x) + f(-x) - f(2x)\| \leq \delta + 2\,\|\,f(0)\,\|\ (\leq 3\delta) \tag{3.6}$$

holds for all $x \in X$.

Substituting $-x$ for x in (3.6), one obtains

$$\|3f(-x) + f(x) - f(-2x)\| \leq \delta + 2\,\|\,f(0)\,\|\ (\leq 3\delta). \tag{3.7}$$

Placing $2^{i-1}x$ on x in (3.6) and (3.7) we get

$$\|A_i(x)\| \leq \delta + 2\,\|\,f(0)\,\|\ (\leq 3\delta),\ \text{and}\ \|B_i(x)\| \leq \delta + 2\,\|\,f(0)\,\|\ (\leq 3\delta) \tag{3.8}$$

for all $i \in N_n$, $n \in N$.

Thus from the formula (3.4), the inequalities (3.5), and the triangle inequality we prove

$$\|S_n(x)\| \leq \sum_{i=1}^{n} \left[|a_i| \|A_i(x)\| + |b_i| \|B_i(x)\| \right]$$

$$(3.9) \qquad \leq (\delta + 2\|f(0)\|) \sum_{i=1}^{n} \left[\frac{2^i+1}{2^{2i+1}} + \frac{2^i-1}{2^{2i+1}} \right] \left(\leq 3\delta \sum_{i=1}^{n} 2^{-i} = 3\delta\left(1-2^{-n}\right) \right)$$

for all $x \in X$ and $n \in N$, completing the proof of this Lemma 3.2. █

In 1998 Jung [14] applied Lemma 3.1 and the author [21] employed Lemma 3.2 with the same absolute bound 3δ instead of our sharper bound δ+2||f(0)||$(\leq \delta + 2\delta = 3\delta)$ in this paper and proved a certain Theorem *on approximately even mappings* f.

Therefore we observe that the following sharper Theorem 3.1 holds.

Theorem 3.1. *Assume an approximately even mapping $f : X \to Y$ satisfies the quadratic inequality* (3.1). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that $||f(0)|| \leq \delta$,which satisfies the quadratic equation* (1.5) *and the inequality*

$$(3.10) \qquad \|f(x) - Q(x)\| \leq \delta + 2\|f(0)\| \ (\leq 3\delta)$$

for a fixed $\delta \geq 0$ and for all $x \in X$.

Note that the right-hand side of (3.10) contains no θ term. In 1998 Jung [14] applied Lemma 3.1 and the author [21] employed Lemma 3.2 with the same absolute bound 3δ instead of our sharper bound δ+2||f(0)||$(\leq \delta + 2\delta = 3\delta)$ in this paper and proved also another Theorem *on approximately odd mappings* f.

Thus we note that the sharper Theorem 3.2 holds.

Theorem 3.2. *Assume an approximately odd mapping $f : X \to Y$ satisfies the additive inequality* (3.1). *Then there exists a unique additive mapping $A : X \to Y$ such that $||f(0)|| \leq \delta$, which satisfies the additive equation* (1.5) *and the inequality*

$$(3.11) \qquad \|f(x) - A(x)\| \leq \delta + 2\|f(0)\| \ (\leq 3\delta)$$

for a fixed $\delta \geq 0$ and for all $x \in X$.

4. STABILITY OF THE EQUATION (1.5) ON A RESTRICTED DOMAIN

In this section, we establish the new Hyers-Ulam stability Theorem 4.1 on a more general restricted domain.

Theorem 4.1. *Let $d > 0$ and $\delta \geq 0$ be fixed. If an approximately even mapping $f : X \to Y$ satisfies the quadratic inequality* (3.1) *for all $x_i \in X$ (i = 1, 2, 3) with $\sum_{i=1}^{3} \|x_i\| \geq d$, then there exists a unique quadratic mapping $Q : X \to Y$, such that $||f(0)|| \leq \delta$ and*

$$(4.1) \qquad \|f(x) - Q(x)\| \leq 4\delta + 3\|f(0)\| \ (\leq 7\delta)$$

for a fixed $\delta \geq 0$ and all $x \in X$.

Proof. Assume $\sum_{i=1}^{3}\|x_i\|< d$. If $x_i = 0$ (i = 1, 2, 3), then we choose a $t\in X$ with $\| t \|\geq 2d$.
Otherwise, choose a $t\in X$ with $\|t\|\geq d$, clearly

(4.2) $$\|x_1-t\|+\|x_2\|+\|x_3+t\|\geq 2\| t \|-\sum_{i=1}^{3}\|x_i\|\geq d\ ,\|x_1\|+\|x_2\|+\| -t \|\geq d\ ,$$
$$\|x_2\|+\|x_3\|+\| t \|\geq d\ ,\ \ \|x_2\|+\| -t \|+\| t \| = 2\| t \|+\|x_2\|\geq d\ .$$

Besides from (3.1) with $x_i = 0$ (i = 1, 2, 3) we get that $\|f(0)\|\leq \delta$.
Therefore from (3.1), (4.2), and the new *functional identity*

$$f\left(\sum_{i=1}^{3}x_i\right)-f(x_1+x_2)-f(x_1+x_3)-f(x_2+x_3)+\sum_{i=1}^{3}f(x_i)+f(0)=$$
$$[f(x_1+x_2+x_3)-f(x_1+x_2-t)-f(x_1+x_3)-f(x_2+x_3+t)+f(x_1-t)+f(x_2)+f(x_3+t)]$$
(with x_1-t on x_1, x_2 on x_2, and x_3+t on x_3)
$$+[f(x_1+x_2-t)-f(x_1+x_2)-f(x_1-t)-f(x_2-t)+f(x_1)+f(x_2)+f(-t)]$$
(with x_1 on x_1, x_2 on x_2, and $-t$ on x_3)
$$+[f(x_2+x_3+t)-f(x_2+x_3)-f(x_2+t)-f(x_3+t)+f(x_2)+f(x_3)+f(t)]$$
(with x_2 on x_1, x_3 on x_2, and t on x_3)
$$-[f(x_2)-f(x_2-t)-f(x_2+t)-f(0)+f(x_2)+f(-t)+f(t)]$$
(with x_2 on x_1, $-t$ on x_2, and t on x_3),

we get
$$\left\|f\left(\sum_{i=1}^{3}x_i\right)-f(x_1+x_2)-f(x_1+x_3)-f(x_2+x_3)+\sum_{i=1}^{3}f(x_i)+f(0)\right\|\leq \delta+\delta+\delta+\delta\ =4\delta,$$
or

(4.3) $$\left\|f\left(\sum_{i=1}^{3}x_i\right)-f(x_1+x_2)-f(x_1+x_3)-f(x_2+x_3)+\sum_{i=1}^{3}f(x_i)\right\|\ \leq 4\delta+\|f(0)\|$$

Applying the Theorem 3.1 and the inequality (4.3), we prove that

$$||f(x)-Q(x)||\leq(4\delta+||f(0)||)+2||f(0)||$$
$$=4\delta+3||f(0)||(\leq 4\delta+3\delta=7\delta).$$

Thus there exists a unique quadratic mapping $Q : X\to Y$ that satisfies the quadratic equation (1.5) and the inequality (4.1), completing the proof of our Theorem 4.1. ▮

Obviously, our inequalities (4.1) and (4.3) are also sharper than the corresponding inequalities of Jung [14], where the right-hand sides were equal to $21\delta(\geq 4\delta+3\|f(0)\|)$ and $7\delta(\geq 4\delta+\|f(0)\|)$, respectively, because $\|f(0)\|\leq\delta$. Similarly, the above-mentioned inequalities (4.1) and (4.3) are also sharper than those inequalities of the author[21], where the right-hand sides were equal to 15δ $(\geq 4\delta+3\|f(0)\|)$ and $5\delta(\geq 4\delta+\|f(0)\|)$, respectively.

We note that if we define $S_3 = \{(x_1,x_2,x_3)\in X^3 : \|x_i\| \langle\, d, i = 1,2,3\}$ for some fixed $d > 0$,
then $\left\{(x_1,x_2,x_3)\in X^3 : \sum_{i=1}^{3}\|x_i\|\geq 3d\right\}\subset X^3\setminus S_3$.

Corollary 4.1. *If we assume that an approximately even mapping* $f : X \to Y$ *satisfies the inequality* (3.1) *for some fixed* $\delta \geq 0$ *and for all* $(x_1, x_2, x_3) \in X^3 \backslash S_3$, *then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (4.1) *for all* $x \in X$.

Corollary 4.2. *An approximately even mapping* $f : X \to Y$ *is quadratic if and only if the following asymptotic condition*

$$\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3) + \sum_{i=1}^{3} f(x_i) \right\| \to 0, \text{ as } \sum_{i=1}^{3} \|x_i\| \to \infty,$$

holds.

Similarly, we prove the following new Theorem 4.2.

Theorem 4.2. *Let* $d > 0$ *and* $\delta \geq 0$ *be fixed. If an approximately odd mapping* $f : X \to Y$ *satisfies the additive inequality* (3.1) *for all* $x_i \in X$ ($i = 1, 2, 3$) *with* $\sum_{i=1}^{3} \|x_i\| \geq d$ *, then there exists a unique additive mapping* $A : X \to Y$, *such that*

(4.4) $$\|f(x) - A(x)\| \leq 4\delta + 3 \, \| \, f(0) \, \| \, (\leq 7\delta)$$

for all $x \in X$.

.

Obviously, our inequalities (4.3) and (4.4) are also sharper than the corresponding inequalities of Jung [14], where the right-hand sides were equal to $7\delta(\geq 4\delta + \|f(0)\|)$ and $21\delta(\geq 4\delta + 3\|f(0)\|)$, respectively, because $\|f(0)\| \leq \delta$. Similarly, the above-mentioned inequalities (4.3) and (4.4) are also sharper than those of the author[21], where the right-hand sides were equal to $5\delta(\geq 4\delta + \|f(0)\|)$ and $15\delta(\geq 4\delta + 3\|f(0)\|)$, respectively.

Corollary 4.3. *If we assume that an approximately odd mapping* $f : X \to Y$ *satisfies the inequality* (3.1) *for some fixed* $\delta \geq 0$ *and for all* $(x_1, x_2, x_3) \in X^3 \backslash S_3$, *then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (4.4) *for all* $x \in X$.

Corollary 4.4. *An approximately odd mapping* $f : X \to Y$ *is additive if and only if the following asymptotic condition*

$$\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3) + \sum_{i=1}^{3} f(x_i) \right\| \to 0, \text{ as } \sum_{i=1}^{3} \|x_i\| \to \infty,$$

holds.

Remark 4.1. From (1.4) for approximately even mappings, the quadratic inequality (1.3) (with $x_1 = x$, $x_2 = x$, $x_3 = -x$), and the triangle inequality, one obtains that

$$4\left\| f(x) - 2^{-2} f(2x) \right\| \leq \|3f(x) + f(-x) - f(2x) - 2f(0)\| + \|-[f(-x) - f(x)]\| + \|2f(0)\|$$
$$\leq \delta + \theta + 2\delta \; = 3\delta + \theta,$$

or

$$\left\| f(x) - 2^{-2} f(2x) \right\| \leq \left(\delta + \frac{\theta}{3}\right)\left(1 - 2^{-2}\right).$$

According to our works [19, 20] on quadratic mappings, one proves that

$$\left\| f(x) - 2^{-2n} f\left(2^n x\right) \right\| \leq \left(\delta + \frac{\theta}{3} \right)\left(1 - 2^{-2n}\right),$$

holds for all $n \in N$, and all $x \in X$, which yields there is a unique quadratic mapping $Q : X \to Y$, such that $Q(x) = \lim_{n\to\infty} 2^{-2n} f\left(2^n x\right)$ and

(4.5) $$\| f(x) - Q(x) \| \leq \delta + \frac{\theta}{3},$$

for all fixed $\delta \geq 0, \theta \geq 0$ and all $x \in X$.

But this inequality is also sharper than the corresponding inequality of Jung [14], where the right-hand side was equal to $\delta + \frac{\theta}{2}$.

Remark 4.2. From the triangle inequality, one gets that

$$||f(-2x)-f(2x)-2[f(-x)-f(x)]|| \leq ||f(-2x)-3f(-x)-f(x)|| + ||-[f(2x)-3f(x)-f(-x)]||$$

$$\leq 2(\delta+2||f(0)||)=2\delta_0 (\leq \ 6\delta),$$

where $\delta_0 = \delta+2||f(0)||$. From this inequality and the hypothesis that the inequality

$$||f(-x)-f(x)|| \leq \theta,$$

holds for all fixed $\theta \geq 0$ and all $x \in X$, as well as from the triangle inequality, we obtain

$$||-2[f(-x)-f(x)]|| \leq ||f(-2x)-f(2x)-2[f(-x)-f(x)]|| + ||-[f(-2x)-f(2x)]||$$

$$\leq 2\delta_0 + \theta,$$

yielding the inequality

$$||f(-x)-f(x)|| \leq \delta_0 + \frac{\theta}{2} = \theta,$$

or

$$\theta = 2\delta_0$$

and the inequality

(4.6) $$||f(-x)-f(x)|| \leq 2\delta_0 = 2(\delta+2||f(0)||)$$

holds for all fixed $\delta \geq 0$, and all $x \in X$.

We observe that the right-hand side $2\delta_0 = 2\delta+4||f(0)||$ of (4.6) contains no θ term.

A faster way to get the inequality (4.6) is by employing the triangle inequality, so that

$$||-2[f(-x)-f(x)]|| \leq ||f(-2x)-3f(-x)-f(x)|| + ||-[f(2x)-3f(x)-f(-x)]||$$

$$+ ||-[f(-2x)-f(2x)]||$$
$$\leq \delta_0 + \delta_0 + \theta = 2\delta_0 + \theta = 4\delta_0,$$

or the inequality (4.6) holds for a fixed $\delta \geq 0$ and all $x \in X$.

From (4.6) and the triangle inequality, one gets that

$$4\left\| f(x) - 2^{-2} f(2x) \right\| \leq \| 3f(x) + f(-x) - f(2x) - 2f(0) \| + \| -[f(-x) - f(x)] \| + \| 2f(0) \|$$

$$\leq \delta+2\delta_0 +2||f(0)||=3\delta_0=3\delta+6||f(0)||,$$

or

$$\left\|f(x)-2^{-2}f(2x)\right\| \leq (\delta+2\,\|\,f(0)\,\|)(1-2^{-2}).$$

According to our works [19, 20] on quadratic mappings, one proves that

$$\left\|f(x)-2^{-2n}f(2^n x)\right\| \leq (\delta+2\,\|\,f(0)\,\|)(1-2^{-2n}),$$

holds for all $n \in N$, and all $x \in X$, which yields there is a unique quadratic mapping $Q : X \to Y$, such that $Q(x)=\lim_{n\to\infty} 2^{-2n} f(2^n x)$ and

(4.7) $$\|f(x)-Q(x)\| \leq \delta+2\,\|\,f(0)\,\|\,(\leq 3\delta)\ .$$

We note that this inequality has right-hand side which is independent of θ and therefore is more interesting than the corresponding inequality of Jung [14] and that of the author [21], which contain a θ term on the right-hand side. In particular, the right hand-side of the inequality of Jung [14] is equal to $\delta+\frac{\theta}{2}$,and that of the author [21] is equal to $\delta+\frac{\theta}{3}(<\delta+\frac{\theta}{2})$.

Remark 4.3. From (1.4) for approximately odd mappings, the additive inequality (3.1) (with $x_1 = x$, $x_2 = x$, $x_3 = -x$), and the triangle inequality, one gets that

$$2\left\|f(x)-2^{-1}f(2x)\right\| \leq \|3f(x)+f(-x)-f(2x)-2f(0)\|+\|-[f(-x)+f(x)]\|+\|2f(0)\|$$
$$\leq \delta+\theta+2\delta\ = 3\delta+\theta,$$

or

$$\left\|f(x)-2^{-1}f(2x)\right\| \leq (3\delta+\theta)(1-2^{-1}),$$

for all fixed $\delta\geq 0,\theta\geq 0$ and all $x\in X$.

According to our works [16-18] on additive mappings, one proves that

$$\left\|f(x)-2^{-n}f(2^n x)\right\| \leq (3\delta+\theta)(1-2^{-n}),$$

holds for all $n \in N$, and all $x \in X$, which yields that there is a unique additive mapping $A : X\to Y$, such that $A(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$ and

(4.8) $$\|f(x)-A(x)\| \leq 3\delta+\theta\ ,$$

for all fixed $\delta\geq 0$, $\theta\geq 0$ and all $x\in X$.

Remark 4.4. From the triangle inequality, one gets that

$$||-4[f(-x)+f(x)]\ || \leq ||f(-2x)-3f(-x)-f(x)||\ +\ ||f(2x)-3f(x)-f(-x)||$$
$$+||-[f(-2x)+f(2x)]\ ||$$
$$\leq (\delta+2||f(0)||)+\ (\delta+2||f(0)||)+\theta,$$

holds for all fixed $\delta\geq 0$, $\theta\geq 0$ and all $x\in X$. From this inequality and the hypothesis that the inequality

$$||f(-x)-f(x)|| \leq \theta,$$

holds for all fixed $\theta \geq 0$ and all $x \in X$, one obtains the inequality

$$\| f(-x)+f(x) \| \leq \frac{\delta}{2} + \|f(0)\| + \frac{\theta}{4} = \theta,$$

or

$$\theta = \frac{2}{3}(\delta + 2\| f(0) \|),$$

and the inequality

$$\| f(-x)+f(x) \| \leq \frac{2}{3}(\delta + 2\| f(0) \|) \ , \tag{4.9}$$

holds for all fixed $\delta \geq 0$, and all $x \in X$.

From (1.4) for approximately odd mappings, the additive inequality (3.1) (with $x_1 = x$, $x_2 = x$, $x_3 = -x$), and the triangle inequality, one gets that

$$2\left\| f(x) - 2^{-1} f(2x) \right\| \leq \| 3f(x) + f(-x) - f(2x) - 2f(0) \| + \| -[f(-x)+f(x)] \| + \| 2f(0) \|$$

$$\leq \delta + \theta + 2\|f(0)\| = \delta + \frac{2}{3}(\delta + 2\| f(0) \|) + 2\|f(0)\|$$

$$= \frac{5}{3}(\delta + 2\|f(0)\|) \ (\leq 5\delta)$$

or

$$\left\| f(x) - 2^{-1} f(2x) \right\| \leq \frac{5}{3}(\delta + 2\| f(0) \|)(1 - 2^{-1}).$$

According to our works [16-18, 22] on additive mappings, one proves that

$$\left\| f(x) - 2^{-n} f(2^n x) \right\| \leq \frac{5}{3}(\delta + 2\| f(0) \|)(1 - 2^{-n}),$$

holds for all $n \in N$, and all $x \in X$, which yields that there is a unique additive mapping $A : X \to Y$, such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ and

$$\| f(x) - A(x) \| \leq \frac{5}{3}(\delta + 2\|f(0)\|) \ , \tag{4.10}$$

for all fixed $\delta \geq 0$, and all $x \in X$.

In the following definition we *generalize* the above functional equation (1.5).

Definition 4.1. A mapping $M : X \to Y$ is called *additive* (resp. *quadratic*) *in* R^4 if M satisfies the functional equation of two types

$$M\left(\sum_{i=1}^{4} x_i\right) + \sum_{1 \leq i < j \leq 4} M(x_i + x_j) = \sum_{i=1}^{4} M(x_i) + \sum_{1 \leq i < j < k \leq 4} M(x_i + x_j + x_k) \tag{4.11}$$

for all $x_i \in X$ (i = 1, 2, 3, 4).

5. Stability of the Equation (4.11)

In this section, we establish new Hyers-Ulam stability Theorems below for new equations.

Theorem 5.1. *Assume an approximately even mapping $f : X \to Y$ satisfies the following quadratic inequality*

$$(5.1)\qquad \left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1 \le i \langle j \le 4} f\left(x_i + x_j\right) - \sum_{i=1}^{4} f\left(x_i\right) - \sum_{1 \le i \langle j \langle k \le 4} f\left(x_i + x_j + x_k\right) \right\| \le \delta ,$$

and the inequality

$$\|f(-x)-f(x)\| \le \theta,$$

for some fixed $\delta \ge 0$ *and* $\theta \ge 0$ *and for all* $x_i \in X$ (i = 1, 2, 3, 4). *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that* $\|f(0)\| \le \delta$, *which satisfies the quadratic equation* (4.11) *and the inequality*

$$(5.2)\qquad \|f(x) - Q(x)\| \le \frac{1}{6}(\delta + 5\| f(0) \| + 5\theta)$$

for all fixed $\delta \ge 0, \theta \ge 0$ *and all* $x \in X$.

Proof. Replacing $x_i = 0$ (i = 1, 2, 3, 4) in (5.1), we find $\|f(0)\| \le \delta$. Thus, substituting $x_i = x$ (i = 1, 2) and $x_j = -x$ (j = 3, 4) in (5.1), one gets

$$\|f(0)+[f(-2x)+f(2x)+4f(0)]-[2f(x)+2f(-x)]-[2f(x)+2f(-x)]\| \le \delta,$$

or

$$(5.3)\qquad \|4f(x) + 4f(-x) - f(2x) - f(-2x)\| \le \delta + 5\| f(0) \|$$

for all fixed $\delta \ge 0$, and all $x \in X$. Therefore from (5.3), (1.4), for approximately even mappings, the quadratic inequality (5.1), and the triangle inequality, we obtain that

$$2\|4f(x) - f(2x)\| \le \|4f(x) + 4f(-x) - f(2x) - f(-2x)\|$$
$$+\|-4[f(-x) - f(x)]\| + \|f(-2x) - f(2x)\| \le \delta + 5\|f(0)\| + 4\theta + \theta = \delta + 5\|f(0)\| + 5\theta,$$

or

$$\left\|f(x) - 2^{-2} f(2x)\right\| \le \left(\frac{1}{6}(\delta + 5\| f(0) \| + 5\theta)\left(1 - 2^{-2}\right)\right) .$$

According to our works [19-20] on quadratic mappings, one proves that

$$(5.4)\qquad \left\|f(x) - 2^{-2n} f\left(2^n x\right)\right\| \le \frac{1}{6}(\delta + 5\| f(0) \| + 5\theta)\left(1 - 2^{-2n}\right),$$

holds for all fixed $\delta \ge 0$, $\theta \ge 0$ and any $n \in N$, and all $x \in X$. Similarly from (4.11) we get, by induction on n, that

$$(5.5)\qquad Q(x) = 2^{-2n} Q\left(2^n x\right),$$

holds for any $n \in N$, and all $x \in X$.

By (5.4), for $n \ge m > 0$, and $h = 2^m x$, we have

$$\left\|2^{-2n} f\left(2^n x\right) - 2^{-2m} f\left(2^m x\right)\right\| = 2^{-2m}\left\|2^{-2(n-m)} f\left(2^{n-m} \cdot 2^m x\right) - f\left(2^m x\right)\right\|$$
$$= 2^{-2m}\left\|2^{-2(n-m)} f\left(2^{n-m} h\right) - f(h)\right\|$$
$$\le 2^{-2m}\frac{1}{6}(\delta + 5\| f(0) \| + 5\theta)\left(1 - 2^{-2(n-m)}\right) = \frac{1}{6}(\delta + 5\| f(0) \| + 5\theta)\left(2^{-2m} - 2^{-2n}\right)$$

$$(5.6) \qquad < \frac{1}{6}(\delta + 5\,\|\,f(0)\,\| + 5\theta)2^{-2m} \to 0 \text{ , as } m \to \infty .$$

From (5.6) and the completeness of Y we get that the Cauchy sequence $\{2^{-2n} f(2^n x)\}$ converges. Therefore we [19-20] may apply a direct method to the definition of Q such that

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$$

holds for all $x \in X$. From the quadratic inequality (5.1), it follows that

$$\left\| Q\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\le i\langle j\le 4} Q(x_i + x_j) - \sum_{i=1}^{4} Q(x_i) - \sum_{1\le i\langle j\langle k\le 4} Q(x_i + x_j + x_k) \right\| \le 2^{-2n}\delta \to 0, \text{ as } n \to \infty ,$$

for all $x_i \in X$ (i = 1, 2, 3, 4). Thus it is obvious that Q satisfies the quadratic equation (4.11). Analogously, by (1.4), we can show that $Q(0) = 0$ (with $x_i = 0$ (i = 1, 2, 3, 4) in (4.11)) and that Q is even from (1.4) with $2^n x$ on place of x,

$$\|Q(x) - Q(-x)\| \le 2^{-2n}\theta \to 0, \text{ as } n \to \infty ,$$

or $Q(-x) = Q(x)$.

According to (5.4), one gets that the inequality (5.2) holds. Assume now that there is another quadratic mapping Q': $X \to Y$ which satisfies the quadratic equation (4.11), the formula (5.5) and the inequality (5.2). Therefore

$$\|Q(x) - Q'(x)\| = 2^{-2n}\left\|Q(2^n x) - Q'(2^n x)\right\| \le 2^{-2n}\left[\left\|Q(2^n x) - f(2^n x)\right\| + \left\|f(2^n x) - Q'(2^n x)\right\|\right]$$

$$\le 2\frac{1}{6}(\delta + 5\,\|\,f(0)\,\| + 5\theta)2^{-2n} \to 0, \text{ as } n \to \infty , \text{ or}$$

$$Q'(x) = Q(x),$$

for all $x_i \in X$, completing the proof of our Theorem 5.1. ∎

We establish below the new Theorem 5.2 on quadratic mappings, which contains no θ terms, and is much sharper than the above Theorem 5.1.

Theorem 5.2. *Assume an approximately even mapping* $f : X \to Y$ *satisfies the following quadratic inequality*

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\le i\langle j\le 4} f(x_i + x_j) - \sum_{i=1}^{4} f(x_i) - \sum_{1\le i\langle j\langle k\le 4} f(x_i + x_j + x_k) \right\| \le \delta ,$$

and the inequality

$$\|4f(x) - f(2x)\| \le \eta,$$

for some fixed $\delta \ge 0$ *and* $\eta \ge 0$ *and for all* $x_i \in X$ (i = 1, 2, 3, 4). *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that* $\|f(0)\| \le \delta$ *, which satisfies the quadratic equation* (4.11) *and the inequality*

$$\|f(x) - Q(x)\| \le \frac{1}{6}(\delta + 5\,\|\,f(0)\,\|)$$

for all fixed $\delta \ge 0$ *and all* $x \in X$.

Proof. From the above inequality (5.3) and the triangle inequality, we get

$$\|4f(x)+4f(-x)-f(2x)-f(-2x)\| \leq \| 4f(x)-f(2x) \| + \| 4f(-x)-f(-2x) \|$$
$$\leq \eta + \eta = 2\eta = \delta + 5 \| f(0) \|,$$

or

$$\eta = \frac{\delta + 5 \| f(0) \|}{2},$$

and therefore

$$||4f(x)-f(2x)|| \leq \frac{\delta + 5 \| f(0) \|}{2},$$

or

$$||f(x)-2^{-2} f(2x) \| \leq \frac{\delta + 5 \| f(0) \|}{6}(1-2^{-2})$$

holds for all fixed $\delta \geq 0$, and all $x \in X$.

The rest of the proof is omitted as similar to the proof of the above Theorem 5.1. ■

We establish below the new stability Theorem 5.3 on additive mappings.

Theorem 5.3. *Assume an approximately odd mapping* $f : X \to Y$ *satisfies the following additive inequality*

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1 \leq i < j \leq 4} f(x_i + x_j) - \sum_{i=1}^{4} f(x_i) - \sum_{1 \leq i < j < k \leq 4} f(x_i + x_j + x_k) \right\| \leq \delta,$$

and the inequalities

$$||f(-x)+f(x)|| \leq \varepsilon,$$

and

$$||2f(x)-f(2x)|| \leq \zeta,$$

for some fixed $\varepsilon \geq 0, \delta \geq 0$ *and* $\zeta \geq 0$ *and for all* $x_i \in X$ (i = 1, 2, 3, 4). *Then there exists a unique additive mapping* $A : X \to Y$, *such that* $\|f(0)\| \leq \delta$ *and*

$$A(x) = \lim_{n \to \infty} 2^{-n} f(2^n x),$$

which satisfies the additive equation (4.11) *and the inequality*

$$\|f(x) - A(x)\| \leq \frac{1}{6}(\delta + 5 \| f(0) \|)$$

for all fixed $\delta \geq 0$ *and all* $x \in X$.

Proof. From the above inequality (5.3) and the triangle inequality, one gets

$$||4[f(-x)+f(x)]|| \leq \|4f(x)+4f(-x)-f(2x)-f(-2x)\|$$

$$+ \quad \|f(-2x)+f(2x)\|,$$

$$\leq (\delta+5\|f(0)\|)+\varepsilon = 4\varepsilon,$$

or

$$\varepsilon=\frac{1}{3}(\delta+5\|f(0)\|) \quad ,$$

for all fixed $\delta \geq 0$.

Therefore

$$\|f(-x)+f(x)\| \leq \frac{1}{3}(\delta+5\|f(0)\|)$$
$$(\leq 2\delta) \ ,$$

for all fixed $\delta \geq 0$ and all $x \in X$.

Thus from the above inequality (5.3) and the triangle inequality, one obtains

$$\|4f(x)+4f(-x)-f(2x)-f(-2x)\| \leq \|2f(x)-f(2x)\|+\|2f(-x)-f(-2x)\|$$

$$+2\|f(-x)+f(x)\|$$

$$\leq \zeta+\zeta+\frac{2}{3}(\delta+5\|f(0)\|) \quad =\delta+5\|f(0)\|,$$

or

$$\zeta=\frac{1}{6}(\delta+5\|f(0)\|),$$

and thus

$$\|2f(x)-f(2x)\| \leq \frac{1}{6}(\delta+5\|f(0)\|) \ (\leq \delta),$$

for all fixed $\delta \geq 0$ and all $x \in X$.

Therefore, we obtain

$$\left\|f(x)-2^{-1}f(2x)\right\| \leq \frac{1}{6}(\delta+5\|f(0)\|)\left(1-2^{-1}\right),$$

for all fixed $\delta \geq 0$ and all $x \in X$.

According to our works [16-18, 22] on additive mappings, one proves that

$$\left\|f(x)-2^{-n}f\left(2^n x\right)\right\| \leq \frac{1}{6}(\delta+5\|f(0)\|)\left(1-2^{-n}\right),$$

holds for all $n \in N$, and all $x \in X$, which yields that there is a unique additive mapping $A : X \to Y,$ such that

$$A(x)=\lim_{n\to\infty} 2^{-n} f\left(2^n x\right)$$

and

$$\|f(x) - A(x)\| \le \frac{1}{6}(\delta + 5\|f(0)\|),$$

for all fixed $\delta \ge 0$,and all $x \in X$.

The rest of the proof is omitted as similar to the proofs of our Theorems above, and in our works [16-18, 22], on additive mappings. ■

6. Stability of the Equation (4.11) On a Restricted Domain

In this section, we establish the new Hyers-Ulam stability Theorem 6.1 for more general quadratic equations on a restricted domain. This Theorem is sharper than our Theorem 6.1 in our work [21].

Theorem 6.1. *Let* $d > 0$, $\delta \ge 0$ *and* $\theta \ge 0$ *be fixed. If an approximately even mapping* $f : X \to Y$ *satisfies the quadratic inequality* (5.1) *for all* $x_i \in X$ (i = 1, 2, 3) *with* $\sum_{i=1}^{4} \|x_i\| \ge d$ *, then there exists a unique quadratic mapping* $Q : X \to Y$, *such that* $\|f(0)\| \le \delta$ *and*

$$\|f(x) - Q(x)\| \le \frac{4\delta + 6\|f(0)\| + 5\theta}{6} \left(\le 5\left(\frac{1}{3}\delta + \frac{\theta}{6}\right)\right) \tag{6.1}$$

for all $x \in X$.

Proof. Assume $\sum_{i=1}^{4} \|x_i\| < d$. We choose a $t \in X$ with $\|t\| \ge 2d$. Clearly, we see

$$\begin{gathered} \|x_1 - t\| + \|x_2\| + \|x_3 + t\| + \|x_4\| \ge 2\|t\| - \sum_{i=1}^{4}\|x_i\| \ge d, \\ \|x_1\| + \|x_2\| + \|x_4\| + \|-t\| = \|t\| + (\|x_1\| + \|x_2\| + \|x_4\|) \ge d, \\ \|x_2\| + \|x_3\| + \|x_4\| + \|t\| \ge d, \ \|x_2\| + \|x_4\| + \|t\| + \|-t\| \ge d. \end{gathered} \tag{6.2}$$

Besides from (5.1) with $x_i = 0$ (i = 1, 2, 3, 4) we get that $\|f(0)\| \le \delta$. Therefore from (5.1), (6.2), and the following new *functional identity*

$$\begin{gathered} f\left(\sum_{i=1}^{4} x_i\right) - f(x_1 + x_2 + x_3) - f(x_1 + x_2 + x_4) - f(x_1 + x_3 + x_4) - f(x_2 + x_3 + x_4) \\ + f(x_1 + x_2) + f(x_1 + x_3) + f(x_1 + x_4) + f(x_2 + x_3) + f(x_2 + x_4) + f(x_3 + x_4) \\ - \sum_{i=1}^{4} f(x_i) - f(0) = \end{gathered}$$

$$\begin{gathered} = f\left(\sum_{i=1}^{4} x_i\right) - f(x_1 + x_2 + x_3) - f(x_1 + x_2 + x_4 - t) - f(x_1 + x_3 + x_4) - f(x_2 + x_3 + x_4 + t) \\ + f(x_1 + x_2 - t) + f(x_1 + x_3) + f(x_1 + x_4 - t) + f(x_2 + x_3 + t) + f(x_2 + x_4) + f(x_3 + x_4 + t) \end{gathered}$$

$$-f(x_1-t)-f(x_2)-f(x_3+t)-f(x_4)$$ (with x_1–t on x_1, x_2 on x_2, x_3+t on x_3,and x_4 on x_4)

$$+[f(x_1+x_2+x_4-t)-f(x_1+x_2+x_4)-f(x_1+x_2-t)-f(x_1+x_4-t)$$
$$-f(x_2+x_4-t)+f(x_1+x_2)+f(x_1+x_4)+f(x_1-t)+f(x_2+x_4)+f(x_2-t)+f(x_4-t)$$
$$-f(x_1)-f(x_2)-f(x_4)-f(-t)]$$ (with x_1 on x_1, x_2 on x_2, x_4 on x_3,and -t on x_4)

$$+[f(x_2+x_3+x_4+t)-f(x_2+x_3+x_4)-f(x_2+x_3+t)-f(x_2+x_4+t)$$
$$-f(x_3+x_4+t)+f(x_2+x_3)+f(x_2+x_4)+f(x_2+t)+f(x_3+x_4)+f(x_3+t)+f(x_4+t)$$
$$-f(x_2)-f(x_3)-f(x_4)-f(t)]$$ (with x_2 on x_1, x_3 on x_2, x_4 on x_3,and t on x_4)

$$-[f(x_2+x_4)-f(x_2+x_4+t)-f(x_2+x_4-t)-f(x_2)-f(x_4)$$
$$+f(x_2+x_4)+f(x_2+t)+f(x_2-t)+f(x_4+t)+f(x_4-t)+f(0)$$
$$-f(x_2)-f(x_4)-f(t)-f(-t)]$$ (with x_2 on x_1, x_4 on x_2, t on x_3,and -t on x_4),

we get the inequality

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\le i\langle j\le 4} f(x_i+x_j) - \sum_{i=1}^{4} f(x_i) - \sum_{1\le i\langle j\langle k\le 4} f(x_i+x_j+x_k)\right\| \le 4\delta + \|f(0)\|. \tag{6.3}$$

Applying the Theorem 5.1 and the inequality (6.3), we prove that

$$\|f(x)-Q(x)\| \le [\,(4\delta+\|f(0)\|)+5\|f(0)\|+5\theta]/6$$
$$=\frac{4\delta+6\|f(0)\|+5\theta}{6}\left(\le 5\left(\frac{1}{3}\delta+\frac{\theta}{6}\right)\right).$$

Thus there exists a unique quadratic mapping $Q : X\to Y$ that satisfies the quadratic equation (4.11) and the inequality (6.1), such that $Q(x)=\lim_{n\to\infty} 2^{-2n} f(2^n x)$, completing the proof of the Theorem 6.1. █

Obviously our inequalities (5.2) and (6.1) are sharper than the corresponding inequalities of the author [21], where the right-hand sides were equal to

$$\delta+\frac{5}{6}\theta \ge \frac{1}{6}(\delta+5\|f(0)\|+5\theta) \quad \textit{and} \quad 5(\delta+\frac{\theta}{6}) \ge \frac{4\delta+6\|f(0)\|+5\theta}{6}\left(\le 5\left(\frac{1}{3}\delta+\frac{\theta}{6}\right)\right),$$

respectively.

We note that if we define $S_4=\{(x_1,x_2,x_3,x_4)\in X^4 : \|x_i\|\langle d, i=1,2,3,4\}$ for some fixed $d>0$, then $\left\{(x_1,x_2,x_3,x_4)\in X^4 : \sum_{i=1}^{4}\|x_i\|\ge 4d\right\}\subset X^4\setminus S_4$.

Corollary 6.1. *If we assume that an approximately even mapping* $f : X\to Y$ *satisfies the inequality* (5.1) *for some fixed* $\delta\ge 0$ *and* $\theta\ge 0$, *and for all* $(x_1, x_2, x_3, x_4)\in X^4\setminus S_4$, *then there exists a unique quadratic mapping* $Q : X\to Y$ *satisfying* (6.1) *for all* $x\in X$.

Corollary 6.2. *An approximately even mapping* $f : X\to Y$ *is quadratic and satisfies the quadratic equation* (4.11) *if and only if the following asymptotic condition*

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1 \le i \langle j \le 4} f\left(x_i + x_j\right) - \sum_{i=1}^{4} f\left(x_i\right) - \sum_{1 \le i \langle j \langle k \le 4} f\left(x_i + x_j + x_k\right) \right\| \to 0,$$

as $\sum_{i=1}^{4} \|x_i\| \to \infty$, *holds.*

We establish below the new Hyers-Ulam stability Theorem 6.2 for general quadratic equations on a restricted domain. This Theorem is sharper than our Theorem 6.1 above.

Theorem 6.2. *Let* $d > 0$, $\delta \ge 0$ *and* $\theta \ge 0$ *be fixed. If an approximately even mapping* $f : X \to Y$ *satisfies the quadratic inequality* (5.1) *for all* $x_i \in X$ $(i = 1, 2, 3)$ *with* $\sum_{i=1}^{4} \|x_i\| \ge d$, *then there exists a unique quadratic mapping* $Q : X \to Y$, *such that* $\|f(0)\| \le \delta$ *and*

(6.4) $$\|f(x) - Q(x)\| \le \frac{2\delta + 3\|f(0)\|}{3} \left(\le \frac{5}{3}\delta\right)$$

for all fixed $\delta \ge 0$ *and all* $x \in X$.

Proof. Applying the Theorem 5.2 and the inequality (6.3), we prove that

$$\|f(x)-Q(x)\| \le \frac{1}{6}[(4\delta + \|f(0)\|) + 5\|f(0)\|]$$
$$= \frac{2\delta + 3\|f(0)\|}{3} \left(\le \frac{5}{3}\delta\right).$$

The rest of the proof of this Theorem is omitted as similar to the proof of the Theorem 6.1. ∎

Corollary 6.3. *If we assume that an approximately even mapping* $f : X \to Y$ *satisfies the inequality* (5.1) *for some fixed* $\delta \ge 0$ *and* $\theta \ge 0$, *and for all* $(x_1, x_2, x_3, x_4) \in X^4 \backslash S_4$, *then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (6.4) *for all* $x \in X$.

Corollary 6.4. *An approximately even mapping* $f : X \to Y$ *is quadratic and satisfies the quadratic equation* (4.11) *if and only if the following asymptotic condition*

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1 \le i \langle j \le 4} f\left(x_i + x_j\right) - \sum_{i=1}^{4} f\left(x_i\right) - \sum_{1 \le i \langle j \langle k \le 4} f\left(x_i + x_j + x_k\right) \right\| \to 0,$$

as $\sum_{i=1}^{4} \|x_i\| \to \infty$, *holds.*

We establish below the new Hyers-Ulam stability Theorem 6.3 for general additive equations on a restricted domain.

Theorem 6.3. *Let $d > 0$, $\delta \geq 0$ and $\theta \geq 0$ be fixed. If an approximately odd mapping $f : X \to Y$ satisfies the additive inequality* (5.1) *for all $x_i \in X$ (i = 1, 2, 3) with $\sum_{i=1}^{4} \|x_i\| \geq d$, then there exists a unique additive mapping $\mathrm{A} : X \to Y$, such that $\|f(0)\| \leq \delta$ and*

$$\|f(x) - Q(x)\| \leq \frac{2\delta + 3\|f(0)\|}{3} \left(\leq \frac{5}{3}\delta\right) \tag{6.5}$$

for all fixed $\delta \geq o$ and all $x \in X$.

Proof. We apply the functional inequality of the above Theorem 6.1,which holds also for the additive mappings, and thus we get the inequality (6.3).

The rest of the proof is omitted as similar to the proof of the Theorem 6.1. ∎

Corollary 6.5. *If we assume that an approximately odd mapping $f : X \to Y$ satisfies the inequality* (5.1) *for some fixed $\delta \geq 0$ and $\theta \geq 0$, and for all $(x_1, x_2, x_3, x_4) \in X^4 \backslash S_4$, then there exists a unique additive mapping $Q : X \to Y$ satisfying* (6.5) *for all $x \in X$.*

Corollary 6.6. *An approximately odd mapping $f : X \to Y$ is additive and satisfies the additive equation* (4.11) *if and only if the following asymptotic condition*

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1 \leq i \langle j \leq 4} f(x_i + x_j) - \sum_{i=1}^{4} f(x_i) - \sum_{1 \leq i \langle j \langle k \leq 4} f(x_i + x_j + x_k) \right\| \to 0,$$

as $\sum_{i=1}^{4} \|x_i\| \to \infty$, *holds.*

REFERENCES

[1] J. ACZEL, *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, 1966.

[2] C. BORELLI and G.L. FORTI, On a general Hyers-Ulam stability result, *Internat. J. Math. Sci.*, **18** (1995), 229-236.

[3] D.G. BOURGIN, Classes of transformations and bordering transformations, *Bull. Amer. Math. Soc.*, **57** (1951), 223-237.

[4] P.W. CHOLEVA, Remarks on the stability of functional equations, *Aequationes Math.*, **27** (1984), 76-86.

[5] ST. CZERWIK, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ Hamburg*, **62** (1992), 59-64.

[6] H. DRLJEVIC, On the stability of the functional quadratic on A-orthogonal vectors, *Publ. Inst. Math.* (*Beograd*) (*N.S.*), **36**(50) (1984), 111-118.

[7] I. FENYO, Osservazioni su alcuni teoremi di D.H. Hyers, *Istit. Lombardo Accad. Sci. Lett. Rend.*, **A 114** (1980), (1982) , 235-242.

[8] I. FENYO, On an inequality of P.W. Cholewa. In: *General Inequalities*, **5**. [Internat. Schriftenreiche Numer. Math., Vol. 80]. Birkhäuser, Basel-Boston, MA, 1987, 277-280.

[9] G.L. FORTI, Hyers-Ulam stability of functional equations in several variables*, Aequationes Math.*, **50** (1995), 143-190.

[10] Z. GAJDA and R. GER, Subadditive multifunctions and Hyers-Ulam stability. In: General Inequalities, **5**.[Internat. Schriftenreiche Numer. Math., Vol. 80]. Birkhäuser, Basel-Boston, MA, 1987.

[11] P. GAVRUTA, An answer to a question of John M. Rassias concerning the stability of Cauchy equation. In: *Advances in Equations and Inequalities*, Hadronic Math. Series, U.S.A., 1999, 67-71.

[12] P.M. GRUBER, Stability of isometries*, Trans. Amer. Math. Soc.,* U.S.A., **245** (1978), 263-277.

[13] D.H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.* **27** (1941), 222-224: The stability of homomorphisms and related topics, "Global Analysis-Analysis on Manifolds", *Teubner - Texte zur Mathematik*, **57** (1983), 140-153.

[14] S.-M. JUNG, On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal.* & *Appl*., **222** (1998), 126-137.

[15] PL. KANNAPPAN, Quadratic functional equation and inner product spaces, *Results Math*., **27** (1995), 368-372.

[16] J.M. RASSIAS, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal*. **46** (1982), 126-130.

[17] J.M. RASSIAS, On approximation of approximately linear mappings by linear mappings, *Bull. Sc. Math*. **108** (1984), 445-446.

[18] J.M. RASSIAS, Solution of a problem of Ulam, *J. Approx. Th*. **57** (1989), 268-273.

[19] J.M. RASSIAS, On the stability of the general Euler-Lagrange functional equation, *Demonstr. Math* 29(1996), 755-766.

[20] J.M. RASSIAS, Solution of the Ulam stability problem for Euler-Lagrange quadratic mappings, *J. Math. Anal. Appl.* **220** (1998), 613-639.

[21] J.M.RASSIAS, On the Ulam stability of mixed type mappings on restricted domains, *J. Math. Anal. Appl.* **276**(2002),747-762

[22] J.M. RASSIAS and M.J. RASSIAS, On the Ulam stability of Jensen and Jensen type mappings on restricted domains, *J. Math. Anal. Appl.* **281**(2003), 516-524.

[23] TH. M. RASSIAS, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc*., **72** (1978), 297-300.

[24] F. SKOF, Sull' approssimazione delle applicazioni localmente δ-additive, *Atti Accad. Sci. Torino Cl Sci. Fis. Mat. Natur*., **117** (1983), 377-389.

[25] S.M. ULAM, *A collection of mathematical problems*, Interscience Publishers, Inc., New York, 1968, p.63; Problems in Modern Mathematics, John Wiley & Sons, Inc, New York, 1964, Chapter VI.